9

Waves in More than One Dimension

Plane Wave Representation in Two and Three Dimensions

Figure 9.1 shows that in two dimensions waves of velocity c may be represented by lines of constant phase propagating in a direction $\mathbf{k}$ which is normal to each line, where the magnitude of $\mathbf{k}$ is the wave number $k = 2\pi/\lambda$.

The direction cosines of $\mathbf{k}$ are given by

$$l = \frac{k_1}{k}, \quad m = \frac{k_2}{k} \quad \text{where} \quad k^2 = k_1^2 + k_2^2$$

and any point $\mathbf{r}(x, y)$ on the line of constant phase satisfies the equation

$$lx + my = p = ct$$

where p is the perpendicular distance from the line to the origin. The displacements at all points $\mathbf{r}(x, y)$ on a given line are in phase and the phase difference ϕ between the origin and a given line is

$$\begin{aligned}\phi = \frac{2\pi}{\lambda}(\text{path difference}) = \frac{2\pi}{\lambda}p &= \mathbf{k} \cdot \mathbf{r} = k_1 x + k_2 y \\ &= kp\end{aligned}$$

Hence, the bracket $(\omega t - \phi) = (\omega t - kx)$ used in a one dimensional wave is replaced by $(\omega t - \mathbf{k} \cdot \mathbf{r})$ in waves of more than one dimension, e.g. we shall use the exponential expression

$$e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}$$

In three dimensions all points $\mathbf{r}(x, y, z)$ in a given wavefront will lie on *planes* of constant phase satisfying the equation

$$lx + my + nz = p = ct$$

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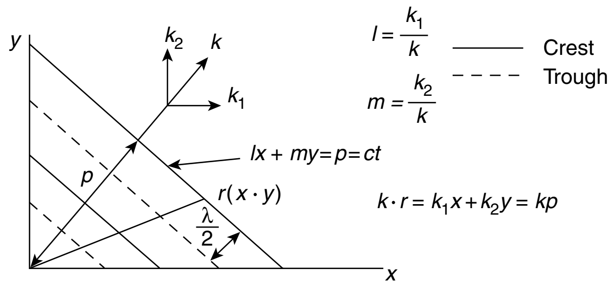


Figure 9.1 Crests and troughs of a two-dimensional plane wave propagating in a general direction **k** (direction cosines l and m). The wave is specified by $lx + my = p = ct$, where p is its perpendicular distance from the origin, travelled in a time t at a velocity c

where the vector **k** which is normal to the plane and in the direction of propagation has direction cosines

$$l = \frac{k_1}{k}, \quad m = \frac{k_2}{k}, \quad n = \frac{k_3}{k}$$

(so that $k^2 = k_1^2 + k_2^2 + k_3^2$) and the perpendicular distance p is given by

$$kp = \mathbf{k} \cdot \mathbf{r} = k_1x + k_2y + k_3z$$

Wave Equation in Two Dimensions

We shall consider waves propagating on a stretched plane membrane of negligible thickness having a mass ρ per unit area and stretched under a uniform tension T. This means that if a line of unit length is drawn in the surface of the membrane, then the material on one side of this line exerts a force T (per unit length) on the material on the other side in a direction perpendicular to that of the line.

If the equilibrium position of the membrane is the xy plane the vibration displacements perpendicular to this plane will be given by z where z depends on the position x, y. In Figure 9.2a where the small rectangular element $ABCD$ of sides δx and δy is vibrating, forces $T\delta x$ and $T\delta y$ are shown acting on the sides in directions which tend to restore the element to its equilibrium position.

In deriving the equation for waves on a string we saw that the tension T along a curved element of string of length dx produced a force perpendicular to x of

$$T\frac{\partial^2 y}{\partial x^2}\,\mathrm{d}x$$

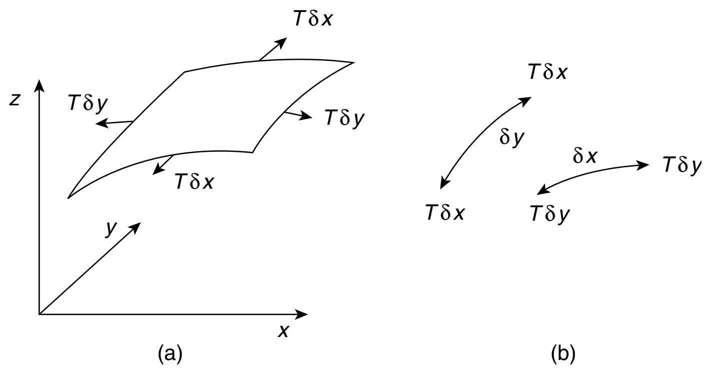


Figure 9.2 Rectangular element of a uniform membrane vibrating in the z-direction subject to one restoring force, $T\delta x$, along its sides of length δy and another, $T\delta y$, along its sides of length δx

where y was the perpendicular displacement. Here in Figure 9.2b by exactly similar arguments we see that a force $T\delta y$ acting on a membrane element of length δx produces a force

$$T\delta y \frac{\partial^2 z}{\partial x^2}\delta x,$$

where z is the perpendicular displacement, whilst another force $T\delta x$ acting on a membrane element of length δy produces a force

$$T\delta x \frac{\partial^2 z}{\partial y^2}\delta y$$

The sum of these restoring forces which act in the z-direction is equal to the mass of the element $\rho\,\delta x\,\delta y$ times its perpendicular acceleration in the z-direction, so that

$$T\frac{\partial^2 z}{\partial x^2}\delta x\,\delta y + T\frac{\partial^2 z}{\partial y^2}\delta x\,\delta y = \rho\,\delta x\,\delta y\frac{\partial^2 y}{\partial t^2}$$

giving the wave equation in two dimensions as

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\rho}{T}\frac{\partial^2 z}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

where

$$c^2 = \frac{T}{\rho}$$

The displacement of waves propagating on this membrane will be given by

$$z = A\,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}\cdot\mathbf{r})} = A\,\mathrm{e}^{\mathrm{i}[\omega t - (k_1 x + k_2 y)]}$$

where

$$k^2 = k_1^2 + k_2^2$$

The reader should verify that this expression for z is indeed a solution to the two-dimensional wave equation when $\omega = ck$.

(Problem 9.1)

Wave Guides

Reflection of a 2D Wave at Rigid Boundaries

Let us first consider a 2D wave propagating in a vector direction $\mathbf{k}(k_1, k_2)$ in the xy plane along a membrane of width b stretched under a tension T between two long rigid rods which present an infinite impedance to the wave.

We see from Figure 9.3 that upon reflection from the line $y = b$ the component k_1 remains unaffected whilst k_2 is reversed to $-k_2$. Reflection at $y = 0$ leaves k_1 unaffected whilst $-k_2$ is reversed to its original value k_2. The wave system on the membrane will therefore be given by the superposition of the incident and reflected waves; that is, by

$$\mathbf{z} = A_1\,\mathrm{e}^{\mathrm{i}[\omega t-(k_1x+k_2y)]} + A_2\,\mathrm{e}^{\mathrm{i}[\omega t-(k_1x-k_2y)]}$$

subject to the boundary conditions that

$$z = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b$$

the positions of the frame of infinite impedance.

The condition $z = 0$ at $y = 0$ requires

$$A_2 = -A_1$$

and $z = 0$ at $y = b$ gives

$$\sin k_2 b = 0$$

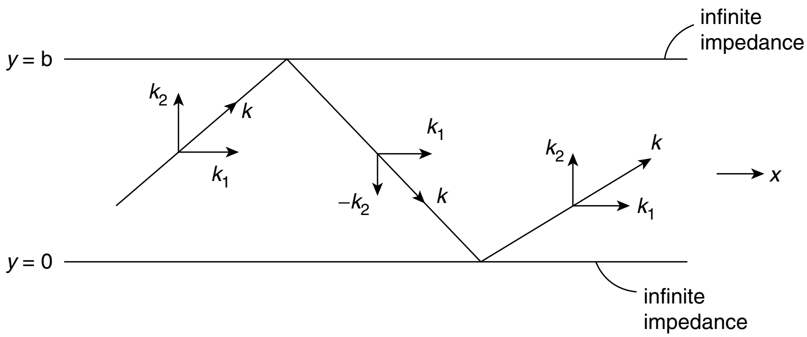


Figure 9.3 Propagation of a two-dimensional wave along a stretched membrane with infinite impedances at $y = 0$ and $y = b$ giving reversal of k_2 at each reflection

or

$$k_2 = \frac{n\pi}{b}$$

(Problem 9.2)

With these values of A_2 and k_2 the displacement of the wave system is given by the real part of **z**, i.e.

$$z = +2A_1 \sin k_2 y \sin(\omega t - k_1 x)$$

which represents a wave travelling along the x direction with a phase velocity

$$v_p = \frac{\omega}{k_1} = \left(\frac{k}{k_1}\right) v$$

where v, the velocity on an infinitely wide membrane, is given by

$$v = \frac{\omega}{k} \quad \text{which is} \quad < v_p$$

because

$$k^2 = k_1^2 + k_2^2$$

Now

$$k^2 = k_1^2 + \frac{n^2\pi^2}{b^2}$$

so

$$k_1 = \left(k^2 - \frac{n^2\pi^2}{b^2}\right)^{1/2} = \left(\frac{\omega^2}{v^2} - \frac{n^2\pi^2}{b^2}\right)^{1/2}$$

and the group velocity for the wave in the x direction

$$v_g = \frac{\partial\omega}{\partial k_1} = \frac{k_1}{\omega}\, v^2 = \left(\frac{k_1}{k}\right) v$$

giving the product

$$v_p v_g = v^2$$

Since k_1 must be real for the wave to propagate we have, from

$$k_1^2 = k^2 - \frac{n^2\pi^2}{b^2}$$

the condition that

$$k^2 = \frac{\omega^2}{v^2} \geq \frac{n^2\pi^2}{b^2}$$

that is

$$\omega \geq \frac{n\pi v}{b}$$

or

$$\nu \geq \frac{nv}{2b},$$

where n defines the mode number in the y direction. Thus, only waves of certain frequencies ν are allowed to propagate along the membrane which acts as a *wave guide*.

There is a cut-off frequency $n\pi v/b$ for each mode of number n and the wave guide acts as a frequency filter (recall the discussion on similar behaviour in wave propagation on the loaded string in Chapter 4). The presence of the $\sin k_2 y$ term in the expression for the displacement z shows that the amplitude varies across the transverse y direction as shown in Figure 9.4 for the mode values $n = 1, 2, 3$. Thus, along any direction in which the waves meet rigid boundaries a standing wave system will be set up analogous to that on a string of fixed length and we shall discuss the implication of this in the section on normal modes and the method of separation of variables.

Wave guides are used for all wave systems, particularly in those with acoustical and electromagnetic applications. Fibre optics is based on wave guide principles, but the major use of wave guides has been with electromagnetic waves in telecommunications. Here the reflecting surfaces are the sides of a copper tube of circular or rectangular cross section. Note that in this case the free space velocity becomes the velocity of light

$$c = \frac{\omega}{k} < v_{\mathrm{p}}$$

the phase velocity, but the relation $v_{\mathrm{p}} v_{\mathrm{g}} = c^2$ ensures that energy in the wave always travels with a group velocity $v_{\mathrm{g}} < c$.

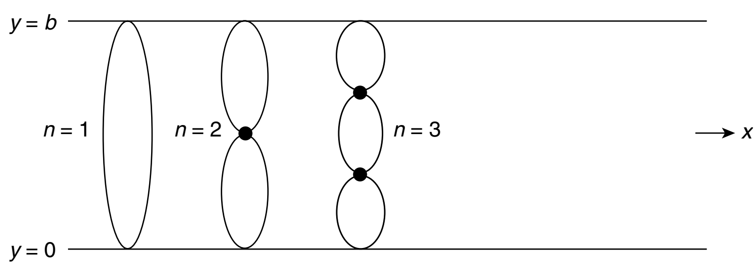


Figure 9.4 Variation of amplitude with y-direction for two-dimensional wave propagating along the membrane of Figure 9.3. Normal modes ($n = 1, 2$ and 3 shown) are set up along any axis bounded by infinite impedances

(Problems 9.3, 9.4, 9.5, 9.6, 9.7, 9.8, 9.9, 9.10, 9.11)

Normal Modes and the Method of Separation of Variables

We have just seen that when waves propagate in more than one dimension a standing wave system will be set up along any axis which is bounded by infinite impedances.

In Chapter 5 we found that standing waves could exist on a string of fixed length l where the displacement was of the form

$$y = A \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} kx \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} \omega_n t,$$

where A is constant and where $\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}$ means that either solution may be used to fit the boundary conditions in space and time. When the string is fixed at both ends, the condition $y = 0$ at $x = 0$ removes the $\cos kx$ solution, and $y = 0$ at $x = l$ requires $k_n l = n\pi$ or $k_n = n\pi/l = 2\pi/\lambda_n$, giving $l = n\lambda_n/2$. Since the wave velocity $c = \nu_n\lambda_n$, this permits frequencies $\omega_n = 2\pi\nu_n = \pi nc/l$, defined as *normal modes of vibration or eigenfrequencies*.

We can obtain this solution in a way which allows us to extend the method to waves in more than one dimension. We have seen that the wave equation

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

has a solution which is the product of two terms, one a function of x only and the other a function of t only.

Let us write $\phi = X(x)T(t)$ and apply the method known as separation of variables.

The wave equation then becomes

$$\frac{\partial^2 X}{\partial x^2}\cdot T = \frac{1}{c^2}X\frac{\partial^2 T}{\partial t^2}$$

or

$$X_{xx}T = \frac{1}{c^2}XT_{tt}$$

where the double subscript refers to double differentiation with respect to the variables. Dividing by $\phi = X(x)T(t)$ we have

$$\frac{X_{xx}}{X} = \frac{1}{c^2}\frac{T_{tt}}{T}$$

where the left-hand side depends on x only and the right-hand side depends on t only. However, both x and t are independent variables and the equality between both sides can only be true when both sides are independent of x and t and are equal to a constant, which we shall take, for convenience, as $-k^2$. Thus

$$\frac{X_{xx}}{X} = -k^2, \quad \text{giving} \quad X_{xx} + k^2 X = 0$$

and

$$\frac{1}{c^2}\frac{T_{tt}}{T} = -k^2, \quad \text{giving} \quad T_{tt} + c^2k^2T = 0$$

$X(x)$ is therefore of the form $\mathrm{e}^{\pm \mathrm{i}kx}$ and $T(t)$ is of the form $\mathrm{e}^{\pm \mathrm{i}ckt}$, so that $\phi = A\,\mathrm{e}^{\pm \mathrm{i}kx}\,\mathrm{e}^{\pm \mathrm{i}ckt}$, where A is constant, and we choose a particular solution in a form already familiar to us by writing

$$\begin{aligned}\phi &= A\,\mathrm{e}^{\mathrm{i}(ckt-kx)}\\ &= A\,\mathrm{e}^{\mathrm{i}(\omega t-kx)}\end{aligned},$$

where $\omega = ck$, or we can write

$$\phi = A \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} kx \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} ckt$$

as above.

Two-Dimensional Case

In extending this method to waves in two dimensions we consider the wave equation in the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

and we write $\phi = X(x)Y(y)T(t)$, where $Y(y)$ is a function of y only.

Differentiating twice and dividing by $\phi = XYT$ gives

$$\frac{X_{xx}}{X} + \frac{Y_{yy}}{Y} = \frac{1}{c^2}\frac{T_{tt}}{T}$$

where the left-hand side depends on x and y only and the right-hand side depends on t only. Since x, y and t are independent variables each side must be equal to a constant, $-k^2$ say. This means that the left-hand side terms in x and y differ by only a constant for all x and y, so that each term is itself equal to a constant. Thus we can write

$$\frac{X_{xx}}{X} = -k_1^2, \quad \frac{Y_{yy}}{Y} = -k_2^2$$

and

$$\frac{1}{c^2}\frac{T_{tt}}{T} = -(k_1^2 + k_2^2) = -k^2$$

giving

$$X_{xx} + k_1^2 X = 0$$
$$Y_{yy} + k_2^2 Y = 0$$
$$T_{tt} + c^2 k^2 T = 0$$

or

$$\phi = A\,\mathrm{e}^{\pm \mathrm{i}k_1 x}\,\mathrm{e}^{\pm \mathrm{i}k_2 y}\,\mathrm{e}^{\pm \mathrm{i}ckt}$$

where $k^2 = k_1^2 + k_2^2$. Typically we may write

$$\phi = A \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_1 x \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_2 y \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} ckt.$$

Three-Dimensional Case

The three-dimensional treatment is merely a further extension. The wave equation is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

with a solution

$$\phi = X(x)Y(y)Z(z)T(t)$$

yielding

$$\phi = A \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_1 x \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_2 y \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_3 z \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} ckt,$$

where $k_1^2 + k_2^2 + k_3^2 = k^2$.

Using vector notation we may write

$$\phi = A\,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}\cdot\mathbf{r})}, \quad \text{where} \quad \mathbf{k}\cdot\mathbf{r} = k_1 x + k_2 y + k_3 z$$

Normal Modes in Two Dimensions on a Rectangular Membrane

Suppose waves proceed in a general direction $\mathbf{k}$ on the rectangular membrane of sides a and b shown in Figure 9.5. Each dotted wave line is separated by a distance $\lambda/2$ and a standing wave system will exist whenever $a = n_1\mathrm{AA}'$ and $b = n_2\mathrm{BB}'$, where n_1 and n_2 are integers.

But

$$\mathrm{AA}' = \frac{\lambda}{2\cos\alpha} = \frac{\lambda}{2}\frac{k}{k_1} = \frac{\lambda}{2}\frac{2\pi}{\lambda}\frac{1}{k_1} = \frac{\pi}{k_1}$$

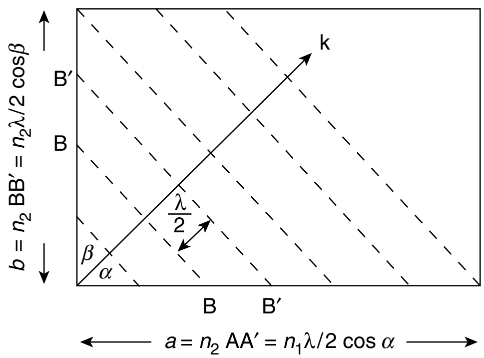


Figure 9.5 Normal modes on a rectangular membrane in a direction **k** satisfying boundary conditions of zero displacement at the edges of length $a = n_1\lambda/2\cos\alpha$ and $b = n_2\lambda/\ 2\cos\beta$

so that

$$a = \frac{n_1\pi}{k_1} \quad \text{and} \quad k_1 = \frac{n_1\pi}{a}.$$

Similarly

$$k_2 = \frac{n_2\pi}{b}$$

Hence

$$k^2 = k_1^2 + k_2^2 = \frac{4\pi^2}{\lambda^2} = \pi^2\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2}\right)$$

or

$$\frac{2}{\lambda} = \sqrt{\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2}}$$

defining the frequency of the n_1th mode on the x-axis and the n_2th mode on the y-axis, that is, the $(n_1 n_2)$ normal mode, as

$$\nu = \frac{c}{2}\sqrt{\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2}}, \quad \text{where} \quad c^2 = \frac{T}{\rho}$$

If **k** is not normal to the direction of either a or b we can write the general solution for the waves as

$$z = A \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_1 x \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_2 y \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} ckt.$$

with the boundary conditions $z = 0$ at $x = 0$ and a; $z = 0$ at $y = 0$ and b.

The condition $z = 0$ at $x = y = 0$ requires a $\sin k_1 x \sin k_2 y$ term, and the condition $z = 0$ at $x = a$ defines $k_1 = n_1\pi/a$. The condition $z = 0$ at $y = b$ gives $k_2 = n_2\pi/b$, so that

$$z = A \sin \frac{n_1 \pi x}{a} \sin \frac{n_2 \pi y}{b} \sin ckt$$

The fundamental vibration is given by $n_1 = 1, n_2 = 1$, so that

$$\nu = \sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\frac{T}{4\rho}}$$

In the general mode $(n_1 n_2)$ zero displacement or nodal lines occur at

$$x = 0, \quad \frac{a}{n_1}, \quad \frac{2a}{n_1}, \ldots a$$

and

$$y = 0, \quad \frac{b}{n_2}, \quad \frac{2b}{n_2}, \ldots b$$

Some of these normal modes are shown in Figure 9.6, where the shaded and plain areas have opposite displacements as shown.

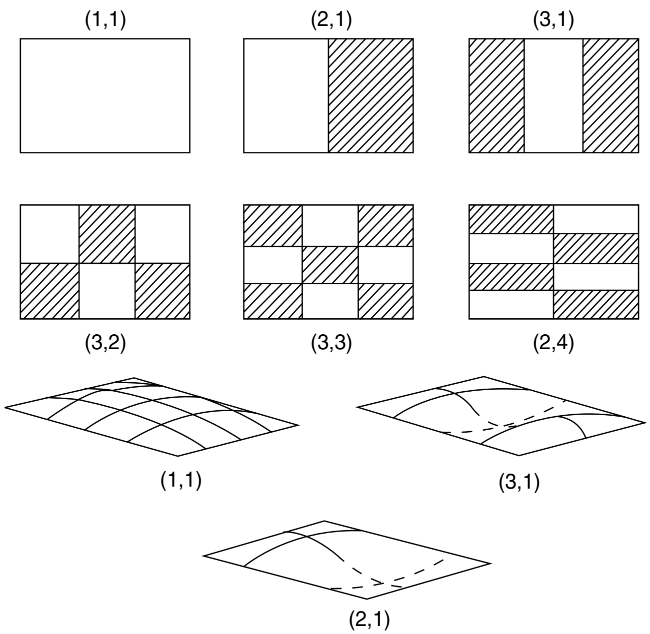


Figure 9.6 Some normal modes on a rectangular membrane with shaded and clear sections having opposite sinusoidal displacements as indicated

The complete solution for a general displacement would be the sum of individual normal modes, as with the simpler case of waves on a string (see the chapter on Fourier Series) where boundary conditions of space and time would have to be met. Several modes of different values $(n_1 n_2)$ may have the same frequency, e.g. in a square membrane the modes (4,7) (7,4) (1,8) and (8,1) all have equal frequencies. If the membrane is rectangular and $a = 3b$, modes (3,3) and (9,1) have equal frequencies.

These modes are then said to be *degenerate*, a term used in describing equal energy levels for electrons in an atom which are described by different quantum numbers.

Normal Modes in Three Dimensions

In three dimensions a normal mode is described by the numbers n_1, n_2, n_3, with a frequency

$$\nu = \frac{c}{2}\sqrt{\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} + \frac{n_3^2}{l_3^2}}, \tag{9.1}$$

where l_1, l_2 and l_3 are the lengths of the sides of the rectangular enclosure. If we now form a rectangular lattice with the x-, y- and z-axes marked off in units of

$$\frac{c}{2l_1}, \quad \frac{c}{2l_2} \quad \text{and} \quad \frac{c}{2l_3}$$

respectively (Figure 9.7), we can consider a vector of components n_1 units in the x-direction, n_2 units in the y-direction and n_3 units in the z-direction to have a length

$$\nu = \frac{c}{2}\sqrt{\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} + \frac{n_3^2}{l_3^2}}$$

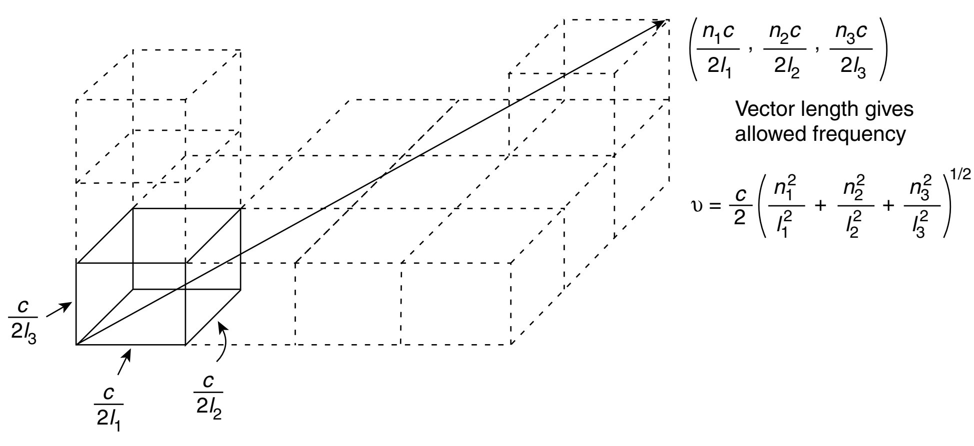


Figure 9.7 Lattice of rectangular cells in frequency space. The length of the vector joining the origin to any cell corner is the value of the frequency of an allowed normal mode. The vector direction gives the propagation direction of that particular mode

Each frequency may thus be represented by a line joining the origin to a point $cn_1/2l_1, cn_2/2l_2, cn_3/2l_3$ in the rectangular lattice.

The length of the line gives the magnitude of the frequency, and the vector direction gives the direction of the standing waves.

Each point will be at the corner of a rectangular unit cell of sides $c/2l_1, c/2l_2$ and $c/2l_3$ with a volume $c^3/8l_1l_2l_3$. There are as many cells as points (i.e. as frequencies) since each cell has eight points at its corners and each point serves as a corner to eight cells.

A very important question now arises: how many normal modes (stationary states in quantum mechanics) can exist in the frequency range ν to $\nu + \mathrm{d}\nu$?

The answer to this question is the total number of all those positive integers n_1, n_2, n_3 for which, from equation (9.1),

$$\nu^2 < \frac{c^2}{4}\left(\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} + \frac{n_3^2}{l_3^2}\right) < (\nu + \mathrm{d}\nu)^2$$

This total is the number of possible points (n_1, n_2, n_3) lying in the positive octant between two concentric spheres of radii ν and $\nu + \mathrm{d}\nu$. The other octants will merely repeat the positive octant values because the n's appear as squared quantities.

Hence the total number of possible points or cells will be

$$\begin{aligned}
&\frac{1}{8}\frac{(\text{volume of spherical shell})}{\text{volume of cell}} \\
&= \frac{4\pi\nu^2\,\mathrm{d}\nu}{8}\cdot\frac{8l_1l_2l_3}{c^3} \\
&= 4\pi l_1l_2l_3\cdot\frac{\nu^2\,\mathrm{d}\nu}{c^3}
\end{aligned}$$

so that the number of possible normal modes in the frequency range ν to $\nu + \mathrm{d}\nu$ *per unit volume* of the enclosure

$$= \frac{4\pi\nu^2\,\mathrm{d}\nu}{c^3}$$

Note that this result, *per unit volume of the enclosure*, is independent of any particular system; we shall consider two very important applications.

Frequency Distribution of Energy Radiated from a Hot Body. Planck's Law

The electromagnetic energy radiated from a hot body at temperature T in the small frequency interval ν to $\nu + \mathrm{d}\nu$ may be written $E_\nu\,\mathrm{d}\nu$. If this quantity is measured experimentally over a wide range of ν a curve T_1 in Figure 9.8 will result. The general shape of the curve is independent of the temperature, but as T is increased the maximum of the curve increases and shifts towards a higher frequency.

The early attempts to describe the shape of this curve were based on two results we have already used.

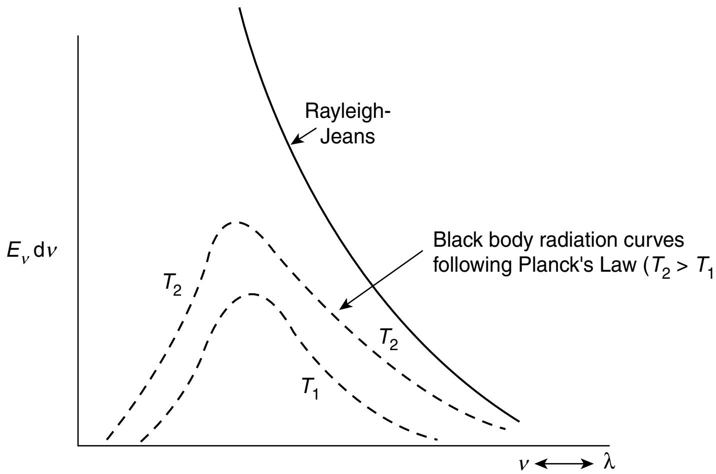


Figure 9.8 Planck's black body radiation curve plotted for two different temperatures $T_2 > T_1$, together with the curve of the classical Rayleigh–0.6-Jeans explanation leading to the 'ultra-violet catastrophe'

In the chapter on coupled oscillations we associated normal modes with 'degrees of freedom', the number of ways in which a system could take up energy. In kinetic theory, assigning an energy $\frac{1}{2}kT$ to each degree of freedom of a monatomic gas at temperature T leads to the gas law $pV = RT = NkT$ where N is Avogadro's number, k is Boltzmann's constant and R is the gas constant.

If we assume that each frequency radiated from a hot body is associated with the normal mode of an oscillator with two degrees of freedom and two transverse planes of polarization, the energy radiated per frequency interval $\mathrm{d}\nu$ may be considered as the product of the number of normal modes or oscillators in the interval $\mathrm{d}\nu$ and an energy contribution of kT from each oscillator for each plane of polarization. This gives

$$E_\nu\,\mathrm{d}\nu = \frac{4\pi\nu^2\,\mathrm{d}\nu\,2kT}{c^3} = \frac{8\pi\nu^2 kT\,\mathrm{d}\nu}{c^3}$$

a result known as the Rayleigh–Jeans Law.

This, however, gives the energy density proportional to ν^2 which, as the solid curve in Figure 9.8 shows, becomes infinite at very high frequencies, a physically absurd result known as the *ultraviolet catastrophe*.

The correct solution to the problem was a major advance in physics. Planck had introduced the quantum theory, which predicted that the average energy value kT should be replaced by the factor $h\nu/(\mathrm{e}^{h\nu/kT} - 1)$, where h is Planck's constant (the unit of action) as shown in Problem 9.12. The experimental curve is thus accurately described by Planck's Radiation Law

$$E_\nu\,\mathrm{d}\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{\mathrm{e}^{h\nu/kT} - 1}\,\mathrm{d}\nu$$

(Problem 9.12)

Debye Theory of Specific Heats

The success of the modern theory of the specific heats of solids owes much to the work of Debye, who considered the thermal vibrations of atoms in a solid lattice in terms of a vast complex of standing waves over a great range of frequencies. This picture corresponds in three dimensions to the problem of atoms spaced along a one dimensional line (Chapter 5). In the specific heat theory each atom was allowed two transverse vibrations (perpendicular planes of polarization) and one longitudinal vibration.

The number of possible modes or oscillations per unit volume in the frequency interval ν to $\nu + \mathrm{d}\nu$ is then given by

$$\mathrm{d}n = 4\pi\nu^2\,\mathrm{d}\nu\left(\frac{2}{c_T^3} + \frac{1}{c_L^3}\right) \tag{9.2}$$

where c_T and c_L are respectively the transverse and longitudinal wave velocities.

Problem 9.12 shows that each mode has an average energy (from Planck's Law) of $\bar{\varepsilon} = h\nu/(\mathrm{e}^{h\nu/kT} - 1)$ and the total energy in the frequency range ν to $\nu + \mathrm{d}\nu$ for a gram atom of the solid of volume V_A is then

$$V_A\bar{\varepsilon}\,\mathrm{d}n = 4\pi V_A\left(\frac{2}{c_T^3} + \frac{1}{c_L^3}\right)\frac{h\nu^3}{\mathrm{e}^{h\nu/kT} - 1}\,\mathrm{d}\nu$$

The total energy per gram atom over all permitted frequencies is then

$$E_A = \int V_A\bar{\varepsilon}\,\mathrm{d}n = 4\pi V_A\left(\frac{2}{c_T^3} + \frac{1}{c_L^3}\right)\int_0^{\nu_m}\frac{h\nu^3}{\mathrm{e}^{h\nu/kT} - 1}\,\mathrm{d}\nu$$

where ν_m is the maximum frequency of the oscillations.

There are N atoms per gram atom of the solid (N is Avogadro's number) and each atom has three allowed oscillation modes, so an approximation to ν_m is found by writing the integral of equation (9.2) for a gram atom as

$$\int \mathrm{d}n = 3N = 4\pi V_A\left(\frac{2}{c_T^3} + \frac{1}{c_L^3}\right)\int_0^{\nu_m}\nu^2\,\mathrm{d}\nu = \frac{4\pi V_A}{3}\left(\frac{2}{c_T^3} + \frac{1}{c_L^3}\right)\nu_m^3$$

The values of c_T and c_L can be calculated from the elastic constants of the solid (see Chapter 6 on longitudinal waves) and ν_m can then be found.

The values of E_A thus becomes

$$E_A = \frac{9N}{\nu_m^3}\int_0^{\nu_m}\frac{h\nu}{\mathrm{e}^{h\nu/kT} - 1}\,\nu^2\,\mathrm{d}\nu$$

and the variation of E_A with the temperature T is the molar specific heat of the substance at constant volume. The specific heat of aluminium calculated by this method is compared with experimental results in Figure 9.9.

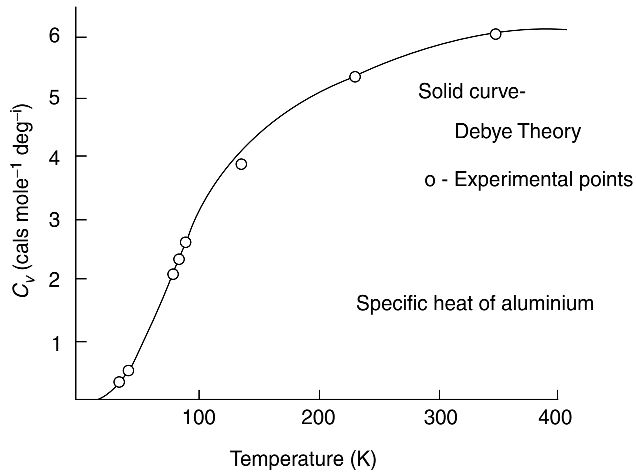


Figure 9.9 Debye theory of specific heat of solids. Experimental values versus theoretical curve for aluminium

(Problems 9.13, 9.14, 9.15, 9.16, 9.17, 9.18, 9.19)

Reflection and Transmission of a Three-Dimensional Wave at a Plane Boundary

To illustrate such an event we choose a physical system of great significance, the passage of a light wave from air to glass. More generally, Figure 9.10 shows a plane polarized electromagnetic wave $\mathbf{E}_\mathrm{i}$ incident at an angle θ to the normal of the plane boundary $z = 0$ separating two dielectrics of impedance Z_1 and Z_2, giving reflected and transmitted rays $\mathbf{E}_\mathrm{r}$ and $\mathbf{E}_\mathrm{t}$, respectively. The boundary condition requires that the tangential electric field E_x is continuous at $z = 0$. The propagation direction $\mathbf{k}_\mathrm{i}$ of $\mathbf{E}_\mathrm{i}$ lies wholly in the plane of the paper $(y = 0)$ but no assumptions are made about the directions of the reflected and transmitted waves (nor about the planes of oscillation of their electric field vectors).

We write

$$\mathbf{E}_\mathrm{i} = A_\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}_\mathrm{i}\cdot\mathbf{r})} = A_\mathrm{i}\,\mathrm{e}^{\mathrm{i}[\omega t - k_\mathrm{i}(x\sin\theta + z\cos\theta)]}$$
$$\mathbf{E}_\mathrm{r} = A_\mathrm{r}\,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}_\mathrm{r}\cdot\mathbf{r})} = A_\mathrm{r}\,\mathrm{e}^{\mathrm{i}[\omega t - (k_{r1}x + k_{r2}y + k_{r3}z)]}$$

and

$$\mathbf{E}_\mathrm{t} = A_\mathrm{t}\,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}_\mathrm{t}\cdot\mathbf{r})} = A_\mathrm{t}\,\mathrm{e}^{\mathrm{i}[\omega t - (k_{t1}x + k_{t2}y + k_{t3}z)]}$$

where $\mathbf{k}_\mathrm{r}(k_\mathrm{r1}, k_\mathrm{r2}, k_\mathrm{r3})$ and $\mathbf{k}_\mathrm{t}(k_\mathrm{t1}, k_\mathrm{t2}, k_\mathrm{t3})$ are respectively the reflected and transmitted propagation vectors.

Since E_x is continuous at $z = 0$ for all x, y, t we have

$$A_\mathrm{i}\,\mathrm{e}^{\mathrm{i}[\omega t - k_\mathrm{i}(x\sin\theta)]} + A_\mathrm{r}\,\mathrm{e}^{\mathrm{i}[\omega t - (k_{r1}x + k_{r2}y)]}$$
$$= A_\mathrm{t}\,\mathrm{e}^{\mathrm{i}[\omega t - (k_{t1}x + k_{t2}y)]}$$

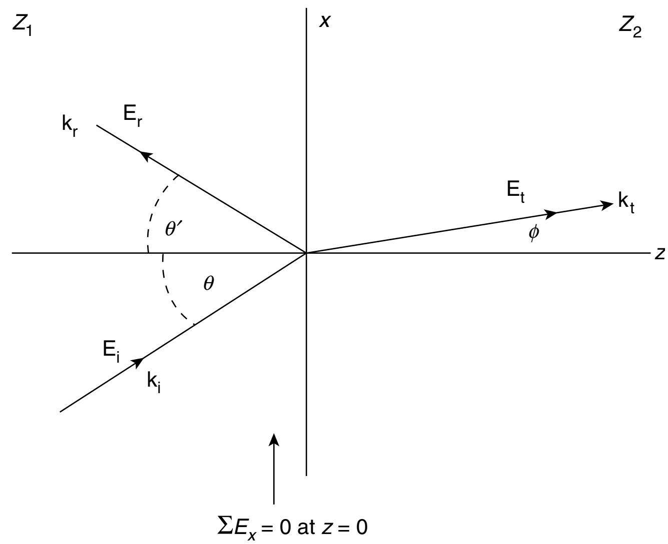


Figure 9.10 Plane-polarized electromagnetic wave propagating in the plane of the paper is represented by vector $\mathbf{E}_i$ and is reflected as vector $\mathbf{E}_r$ and transmitted as vector $\mathbf{E}_t$ at a plane interface between media of impedances Z_1 and Z_2. No assumptions are made about the planes of propagation of $\mathbf{E}_r$ and $\mathbf{E}_t$. From the boundary condition that the electric field component E_x is continuous at the plane $z = 0$ it follows that (1) vectors $\mathbf{E}_i\,\mathbf{E}_r$ and $\mathbf{E}_t$ propagate in the same plane; (2) $\theta = \theta'$ (angle of incidence = angle of reflection); (3) Snell's law $(\sin\theta/\sin\phi) = n_2/n_1$, where n is the refractive index

an identity which is only possible if the indices of all three terms are identical; that is

$$\begin{aligned}\omega t - k_i x \sin\theta &\equiv \omega t - k_{r1}x + k_{r2}y \\ &\equiv \omega t - k_{t1}x + k_{t2}y\end{aligned}$$

Equating the coefficients of x in this identity gives

$$k_i \sin\theta = k_{r1} = k_{t1}$$

whilst equal coefficients of y give

$$0 = k_{r2} = k_{t2}$$

The relation

$$k_{r2} = k_{t2} = 0$$

shows that the reflected and transmitted rays have no component in the y direction and lie wholly in the xz plane of incidence; that is, incident reflected and transmitted (refracted) rays are coplanar.

Now the magnitude

$$k_{\mathrm{i}} = k_{\mathrm{r}} = \frac{2\pi}{\lambda_1}$$

since both incident and reflected waves are travelling in medium Z_1. Hence

$$k_{\mathrm{i}} \sin\theta = k_{\mathrm{r}1}$$

gives

$$k_{\mathrm{i}} \sin\theta = k_{\mathrm{r}} \sin\theta'$$

that is

$$\theta = \theta'$$

so the angle of incidence equals the angle of reflection.

The magnitude

$$k_{\mathrm{t}} = \frac{2\pi}{\lambda_2}$$

so that

$$k_{\mathrm{i}} \sin\theta = k_{\mathrm{t}1} = k_{\mathrm{t}} \sin\phi$$

gives

$$\frac{2\pi}{\lambda_1}\sin\theta = \frac{2\pi}{\lambda_2}\sin\phi$$

or

$$\frac{\sin\theta}{\sin\phi} = \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1}\left[\frac{\text{Refractive Index (medium 2)}}{\text{Refractive Index (medium 1)}}\right]$$

a relationship between the angles of incidence and refraction which is well known as Snell's Law.

Total Internal Reflection and Evanescent Waves

On p. 254 we discussed the propagation of an electromagnetic wave across the boundary between air and a dielectric (glass, say). We now consider the reverse process where a wave in the dielectric crosses the interface into air.

Snell's Law still holds so we have, in Figure 9.11,

$$n_1 \sin\theta = n_2 \sin\phi$$

where

$$n_1 > n_2 \quad \text{and} \quad n_2/n_1 = n_r < 1$$

Thus

$$\sin\theta = (n_2/n_1)\sin\phi = n_r \sin\phi$$

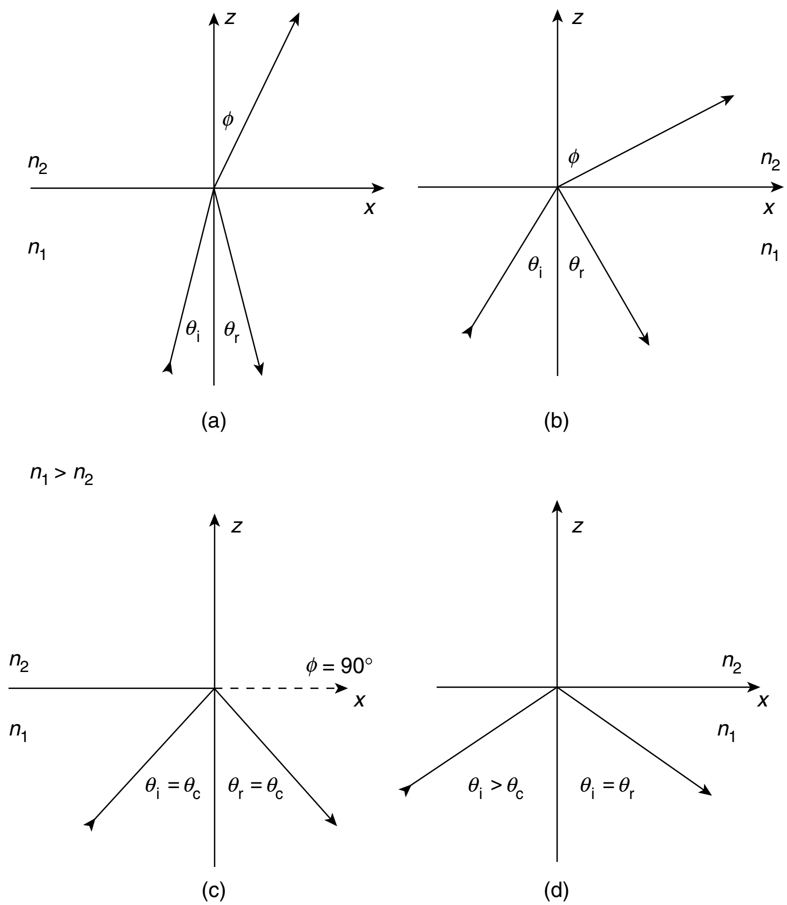


Figure 9.11 When light propagates from a dense to a rare medium $(n_1 > n_2)$ Snell's Law defines $\theta = \theta_c$ as that angle of incidence for which $\phi = 90^\circ$ and the refracted ray is tangential to the plane boundary. Total internal reflection can take place but the boundary conditions still require a transmitted wave known as the evanescent or surface wave. It propagates in the *x* direction but its amplitude decays exponentially with *z*

with $\phi > \theta$. Eventually a critical angle of incidence θ_c is reached where $\phi = 90^\circ$ and $\sin\theta = n_r$; for $\theta > \theta_c$, $\sin\theta > n_r$. For glass to air $n_r = \frac{1}{1.5}$ and $\theta_c = 42^\circ$.

It is evident that for $\theta \geq \theta_c$ no electromagnetic energy is transmitted into the rarer medium and the incident wave is said to suffer *total internal reflection*.

In the reflection coefficients $R_{||}$ and $R_{\perp}$ on p. 218 we may replace $\cos\phi$ by

$$(1 - \sin^2\phi)^{1/2} = [1 - (\sin\theta/n_r)^2]^{1/2}$$

and rewrite

$$R_{||} = \frac{(n_{\mathrm{r}}^2 - \sin^2\theta)^{1/2} - n_{\mathrm{r}}^2\cos\theta}{(n_{\mathrm{r}}^2 - \sin^2\theta)^{1/2} + n_{\mathrm{r}}^2\cos\theta}$$

and

$$R_{\perp} = \frac{\cos\theta - (n_{\mathrm{r}}^2 - \sin^2\theta)^{1/2}}{\cos\theta + (n_{\mathrm{r}}^2 - \sin^2\theta)^{1/2}}$$

Now for $\theta > \theta_{\mathrm{c}}$, $\sin\theta > n_{\mathrm{r}}$ and the bracketed quantities in $R_{||}$ and $R_{\perp}$ are negative so that $R_{||}$ and $R_{\perp}$ are complex quantities; that is $(E_{\mathrm{r}})_{||}$ and $(E_{\mathrm{r}})_{\perp}$ have a phase relation which depends on θ.

It is easily checked that the product of R and R^* is unity so we have $R_{||}R_{||}^* = R_{\perp}R_{\perp}^* = 1$. This means, for both the examples of Figure 8.8, that the incident and reflected intensities I_{i} and $I_{\mathrm{r}} = 1$. The transmitted intensity $I_{\mathrm{t}} = 0$ so that no energy is carried across the boundary.

However, if there is no transmitted wave we cannot satisfy the boundary condition $E_{\mathrm{i}} + E_{\mathrm{r}} = E_{\mathrm{t}}$ on p. 254, using only incident and reflected waves. We must therefore assert that a transmitted wave does exist but that it cannot on the average carry energy across the boundary.

We now examine the implications of this assertion, using Figure 9.10 above, and we keep the notation of p. 254. This gives a transmitted electric field vector

$$\begin{aligned} E_{\mathrm{t}} &= A_{\mathrm{t}}\,\mathrm{e}^{\mathrm{i}[\omega t - (k_{\mathrm{t1}}x + k_{\mathrm{t2}}y + k_{\mathrm{t3}}z)]} \\ &= A_{\mathrm{t}}\,\mathrm{e}^{\mathrm{i}[\omega t - k_{\mathrm{t}}(x\sin\phi + z\cos\phi)]} \end{aligned}$$

because $y = 0$ in the xz plane, $k_{\mathrm{t1}} = k_{\mathrm{t}}\sin\phi$ and $k_{\mathrm{t3}} = k_{\mathrm{t}}\cos\phi$. Now

$$\begin{aligned} \cos\phi =& 1 - \sin^2\phi = 1 - \sin^2\theta/n_{\mathrm{r}}^2 \\ & \therefore k_{\mathrm{t}}\cos\phi = \pm k_{\mathrm{t}}(1 - \sin^2\theta/n_{\mathrm{r}}^2)^{1/2} \end{aligned}$$

which for $\theta > \theta_{\mathrm{c}}$ gives $\sin\theta > n_r$ so that

$$k_{\mathrm{t}}\cos\phi = \mp \mathrm{i}k_{\mathrm{t}}\left(\frac{\sin^2\theta}{n_{\mathrm{r}}^2} - 1\right)^{1/2} = \mp\mathrm{i}\beta$$

We also have

$$k_{\mathrm{t}}\sin\phi = k_{\mathrm{t}}\sin\theta/n_{\mathrm{r}}$$

so

$$E_{\mathrm{t}} = A_{\mathrm{t}}\,\mathrm{e}^{\mp\beta z}\,\mathrm{e}^{\mathrm{i}(\omega t - k_{\mathrm{r}}x\sin\theta/n_{\mathrm{r}})}$$

The alternative factor $e^{+\beta z}$ defines an exponential growth of A_t which is physically untenable and we are left with a wave whose amplitude decays exponentially as it penetrates the less dense medium. The disturbance travels in the x direction along the interface and is known as a *surface* or *evanescent wave*.

It is possible to show from the expressions for $R_{||}$ and $R_{\perp}$ on p. 258 that except at $\theta = 90°$ the incident and the reflected electric field components for $(E)_{||}$ in one case and $(E)_{\perp}$ in the other, do not differ in phase by π rad and cannot therefore cancel each other out. The continuity of the tangential component of **E** at the boundary therefore leaves a component parallel to the interface which propagates as the surface wave. This effect has been observed at optical frequencies.

Moreover, if only a very thin air gap exists between two glass blocks it is possible for energy to flow across the gap allowing the wave to propagate in the second glass block. This process is called *frustrated total internal reflection* and has its quantum mechanical analogue in the tunnelling effect discussed on p. 431.

Problem 9.1

Show that

$$z = A\,e^{i\{\omega t - (k_1 x + k_2 y)\}}$$

where $k^2 = \omega^2/c^2 = k_1^2 + k_2^2$ is a solution of the two-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

Problem 9.2

Show that if the displacement of the waves on the membrane of width b of Figure 9.3 is given by the superposition

$$\mathbf{z} = A_1\,e^{i[\omega t - (k_1 x + k_2 y)]} + A_2\,e^{i[\omega t - (k_1 x - k_2 y)]}$$

with the boundary conditions

$$z = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b$$

then the real part of **z** is

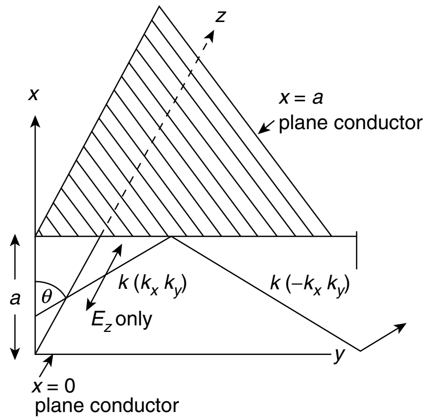
$$z = +2A_1 \sin k_2 y \sin(\omega t - k_1 x)$$

where

$$k_2 = \frac{n\pi}{b}$$

Problem 9.3

An electromagnetic wave loses negligible energy when reflected from a highly conducting surface. With repeated reflections it may travel along a transmission line or wave guide consisting of two parallel, infinitely conducting planes (separation a). If the wave



is plane polarized, so that only E_z exists, then the propagating direction $\mathbf{k}$ lies wholly in the xy plane. The boundary conditions require that the total tangential electric field E_z is zero at the conducting surfaces $x = 0$ and $x = a$. Show that the first boundary condition allows E_z to be written $E_z = E_0(\mathrm{e}^{\mathrm{i}k_x x} - \mathrm{e}^{-\mathrm{i}k_x x})\,\mathrm{e}^{\mathrm{i}(k_y y - \omega t)}$, where $k_x = k\cos\theta$ and $k_y = k\sin\theta$ and the second boundary condition requires $k_x = n\pi/a$.

If $\lambda_0 = 2\pi c/\omega$, $\lambda_c = 2\pi/k_x$ and $\lambda_g = 2\pi/k_y$ are the wavelengths propagating in the x and y directions respectively show that

$$\frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2}$$

We see that for $n = 1$, $k_x = \pi/a$ and $\lambda_c = 2a$, and that as ω decreases and λ_0 increases, $k_y = k\sin\theta$ becomes imaginary and the wave is damped. Thus, $n = 2(k_x = 2\pi/a)$ gives $\lambda_c = a$, the 'critical wavelength', i.e. the longest wavelength propagated by a waveguide of separation a. Such cut-off wavelengths and frequencies are a feature of wave propagation in periodic structures, transmission lines and wave-guides.

Problem 9.4

Show, from equations (8.1) and (8.2), that the magnetic field in the plane-polarized electromagnetic wave of Problem 9.3 has components in both x- and y-directions. [When an electromagnetic wave propagating in a waveguide has only transverse electric field vectors and no electric field in the direction of propagation it is called a transverse electric (TE) wave. Similarly a transverse magnetic (TM) wave may exist. The plane-polarized wave of Problem 9.3 is a transverse electric wave; the corresponding transverse magnetic wave would have H_z, E_x and E_y components. The values of n in Problem 9.3 satisfying the boundary conditions are written as subscripts to define the exact mode of propagation, e.g. TE_{10}.]

Problem 9.5

Use the value of the inductance and capacitance of a pair of plane parallel conductors of separation a and width b to show that the characteristic impedance of such a waveguide is given by

$$Z_0 = \frac{a}{b}\sqrt{\frac{\mu}{\varepsilon}}\,\Omega$$

where μ and ε are respectively the permeability and permittivity of the medium between the conductors.

Problem 9.6

Consider either the Poynting vector or the energy per unit volume of an electromagnetic wave to show that the power transmitted by a single positive travelling wave in the waveguide of Problem 9.5 is $\frac{1}{2}abE_0^2\sqrt{\varepsilon/\mu}$.

Problem 9.7

An electromagnetic wave (**E**, **H**) propagates in the x-direction down a perfectly conducting hollow tube of arbitrary cross section. The tangential component of **E** at the conducting walls must be zero at all times.

Show that the solution $\mathbf{E} = E(y, z)\ \mathbf{n} \cos(\omega t - k_x x)$ substituted in the wave equation yields

$$\frac{\partial^2 E(y,z)}{\partial y^2} + \frac{\partial^2 E(y,z)}{\partial z^2} = -k^2 E(y,z),$$

where $k^2 = \omega^2/c^2 - k_x^2$ and k_x is the wave number appropriate to the x-direction, **n** is the unit vector in any direction in the (y, z) plane.

Problem 9.8

If the waveguide of Problem 9.7 is of rectangular cross-section of width a in the y-direction and height b in the z-direction, show that the boundary conditions $E_x = 0$ at $y = 0$ and a and at $z = 0$ and b in the wave equation of Problem 9.7 gives

$$E_x = A \sin\frac{m\pi y}{a} \sin\frac{n\pi z}{b} \cos(\omega t - k_x x),$$

where

$$k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

Problem 9.9

Show, from Problems 9.7 and 9.8, that the lowest possible value of ω (the cut-off frequency) for k_x to be real is given by $m = n = 1$.

Problem 9.10

Prove that the product of the phase and group velocity $\omega/k_x, \partial\omega/\partial k_x$ of the wave of Problems 9.7–9.9 is c^2, where c is the velocity of light.

Problem 9.11

Consider now the extension of Problem 9.2 where the waves are reflected at the rigid edges of the rectangular membrane of sides length a and b as shown in the diagram. The final displacement is the result of the superposition

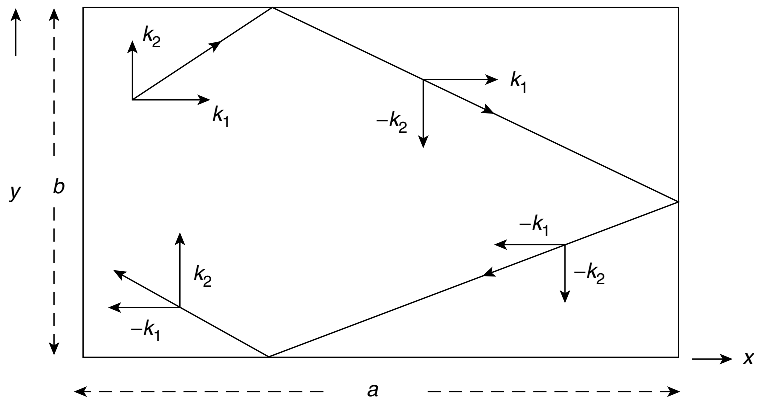
$$\begin{aligned}\mathbf{z} = &A_1\, e^{i[\omega t - (k_1 x + k_2 y)]} \\ &+ A_2\, e^{i[\omega t - (k_1 x - k_2 y)]} \\ &+ A_3\, e^{i[\omega t - (-k_1 x - k_2 y)]} \\ &+ A_4\, e^{i[\omega t - (-k_1 x + k_2 y)]}\end{aligned}$$

with the boundary conditions

$$z = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a$$

and

$$z = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b$$



Show that this leads to a displacement

$$z = -4A_1 \sin k_1 x \sin k_2 y \cos \omega t$$

(the real part of **z**), where

$$k_1 = \frac{n_1 \pi}{a} \quad \text{and} \quad k_2 = \frac{n_2 \pi}{b}$$

Problem 9.12

In deriving the result that the average energy of an oscillator at frequency ν and temperature T is given by

$$\bar{\varepsilon} = \frac{h\nu}{e^{(h\nu/kT)} - 1}$$

Planck assumed that a large number N of oscillators had energies $0, h\nu, 2h\nu \dots nh\nu$ distributed according to Boltzmann's Law

$$N_n = N_0 \, e^{-nh\nu/kT}$$

where the number of oscillators N_n with energy $nh\nu$ decreases exponentially with increasing n.

Use the geometric progression series

$$N = \sum_n N_n = N_0(1 + e^{-h\nu/kT} + e^{-2h\nu/kT} \dots)$$

to show that

$$N = \frac{N_0}{1 - e^{-h\nu/kT}}$$

If the total energy of the oscillators in the nth energy state is given by

$$E_n = N_n nh\nu$$

prove that the total energy over all the n energy states is given by

$$E = \sum_n E_n = N_0 \frac{h\nu\, \mathrm{e}^{-h\nu/kT}}{(1 - \mathrm{e}^{-h\nu/kT})^2}$$

Hence show that the average energy per oscillator

$$\bar{\varepsilon} = \frac{E}{N} = \frac{h\nu}{\mathrm{e}^{h\nu/kT} - 1}$$

and expand the denominator to show that for $h\nu \ll kT$, that is low frequencies and long wavelengths. Planck's Law becomes the classical expression of Rayleigh–Jeans.

Problem 9.13

The wave representation of a particle, e.g. an electron, in a rectangular potential well throughout which $V = 0$ is given by Schrödinger's time-independent equation

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = -\frac{8\pi^2 m}{h^2} E\Psi,$$

where E is the particle energy, m is the mass and h is Planck's constant. The boundary conditions to be satisfied are $\psi = 0$ at $x = y = z = 0$ and at $x = L_x, y = L_y, z = L_z$, where L_x, L_y and L_z are the dimensions of the well.

Show that

$$\Psi = A \sin \frac{l\pi x}{L_x} \sin \frac{r\pi y}{L_y} \sin \frac{n\pi z}{L_z}$$

is a solution of Schrödinger's equation, giving

$$E = \frac{h^2}{8m}\left(\frac{l^2}{L_x^2} + \frac{r^2}{L_y^2} + \frac{n^2}{L_z^2}\right)$$

When the potential well is cubical of side L,

$$E = \frac{h^2}{8mL^2}(l^2 + r^2 + n^2)$$

and the lowest value of the quantized energy is given by

$$E = E_0 \quad \text{for} \quad l = 1, \quad r = n = 0$$

Show that the next energy levels are $3E_0, 6E_0$ (three-fold degenerate), $9E_0$ (three-fold degenerate), $11E_0$ (three-fold degenerate), $12E_0$ and $14E_0$ (six-fold degenerate).

Problem 9.14

Show that at low energy levels (long wavelengths) $h\nu \ll kT$, Planck's radiation law is equivalent to the Rayleigh–Jeans expression.

Problem 9.15

Planck's radiation law, expressed in terms of energy per unit range of wavelength instead of frequency, becomes

$$E_\lambda = \frac{8\pi ch}{\lambda^5 (e^{ch/\lambda kT} - 1)}$$

Use the variable $x = ch/\lambda kT$ to show that the total energy per unit volume at temperature $T°$ absolute is given by

$$\int_0^\infty E_\lambda \, d\lambda = aT^4 \, \mathrm{J\,m^{-3}}$$

where

$$a = \frac{8\pi^5 k^4}{15c^3h^3}$$

(The constant $ca/4 = \sigma$, Stefan's Constant in the Stefan-Boltzmann Law.) Note that

$$\int_0^\infty \frac{x^3 \, dx}{e^x - 1} = \frac{\pi^4}{15}$$

Problem 9.16

Show that the wavelength λ_m at which E_λ in Problem 9.15 is a maximum is given by the solution of

$$\left(1 - \frac{x}{5}\right) e^x = 1, \quad \text{where} \quad x = \frac{ch}{\lambda kT}$$

The solution is $ch/\lambda_m kT = 4.965$.

Problem 9.17

Given that $ch/\lambda_m = 5\,kT$ in Problem 9.16, show that if the sun's temperature is about 6000 K, then $\lambda_m \approx 4.7 \times 10^{-7}$ m, the green region of the visible spectrum where the human eye is most sensitive (evolution ?).

Problem 9.18

The tungsten filament of an electric light bulb has a temperature of ≈ 2000 K. Show that in this case $\lambda_m \approx 14 \times 10^{-7}$ m, well into the infrared. Such a lamp is therefore a good heat source but an inefficient light source.

Problem 9.19

A free electron (travelling in a region of zero potential) has an energy

$$E = \frac{p^2}{2m} = \left(\frac{\hbar^2}{2m}\right) k^2 = E(k)$$

where the wavelength

$$\lambda = h/p = 2\pi/k$$

In a weak periodic potential; for example, in a solid which is a good electrical conductor, $E = (\hbar^2/2m^*)\,k^2$, where m^* is called the effective mass. (For valence electrons $m^* \approx m$.)

Represented as waves, the electrons in a cubic potential well $(V = 0)$ of side L have allowed wave numbers k, where

$$k^2 = k_x^2 + k_y^2 + k_z^2 \quad \text{and} \quad k_i = \frac{n_i \pi}{L}$$

(see Problem 9.13). For each value of k there are two allowed states (each defining the spin state of the single occupying electron–Pauli's principle). Use the arguments in Chapter 9 to show that the total number of states in k space between the values k and $k + \mathrm{d}k$ is given by

$$P(k) = 2\left(\frac{L}{\pi}\right)^3 \frac{4\pi k^2\,\mathrm{d}k}{8}$$

Use the expression $E = (\hbar^2/2m^*)\,k^2$ to convert this into the number of states $S(E)\,\mathrm{d}E$ in the energy interval $\mathrm{d}E$ to give

$$S(E) = \frac{A}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}$$

where $A = L^3$.

If there are N electrons in the N lowest energy states consistent with Pauli's principle, show that the integral

$$\int_0^{E_f} S(E)\,\mathrm{d}E = N$$

gives the Fermi energy level

$$E_{\mathrm{f}} = \frac{\hbar^2}{2m^*}\left(\frac{3\pi^2 N}{A}\right)^{2/3}$$

where E_{f} is the kinetic energy of the most energetic electron when the solid is in its ground state.

Summary of Important Results

Wave Equation in Two Dimensions

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

Lines of constant phase $lx + my = ct$ propagate in direction $\mathbf{k}(k_1, k_2)$ where $l = k_1/k, m = k_2/k, k^2 = k_1^2 + k_2^2$ and $c^2 = \omega^2/k^2$. Solution is

$$z = A\,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}\cdot\mathbf{r})} \quad \text{for} \quad \mathbf{r}(x, y)$$

where $k \cdot r = k_1 x + k_2 y$.

Wave Equation in Three Dimensions

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

Planes of constant phase $lx + my + nz = ct$ propagate in a direction

$$\mathbf{k}(k_1, k_2, k_3), \quad \text{where} \quad l = k_1/k, \quad m = k_2/k, \quad n = k_3/k$$
$$k^2 = k_1^2 + k_2^2 + k_3^2 \quad \text{and} \quad c^2 = \omega^2/k^2.$$

Solution is

$$\phi = A\,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}\cdot\mathbf{r})} \quad \text{for} \quad \mathbf{r}(x, y, z).$$

Wave Guides

Reflection from walls $y = 0, y = b$ in a two-dimensional wave guide for a wave of frequency ω and vector direction $\mathbf{k}(k_1, k_2)$ gives normal modes in the y direction with $k_2 = n\pi/b$ and propagation in the x direction with phase velocity

$$v_p = \frac{\omega}{k_1} > \frac{\omega}{k} = v$$

and group velocity

$$v_g = \frac{\partial\omega}{\partial k_1} \quad \text{such that} \quad v_p v_g = v^2$$

Cut-off frequency

Only frequencies $\omega \geq n\pi v/b$ will propagate where n is mode number.

Normal Modes in Three Dimensions

Wave equation separates into three equations (one for each variable x, y, z) to give solution

$$= A \begin{matrix}\sin\\ \cos\end{matrix} k_1 x \begin{matrix}\sin\\ \cos\end{matrix} k_2 y \begin{matrix}\sin\\ \cos\end{matrix} k_3 z \begin{matrix}\sin\\ \cos\end{matrix} \omega t$$

(Boundary conditions determine final form of solution.)

For waves of velocity c, the number of normal modes per unit volume of an enclosure in the frequency range ν to $\nu + \mathrm{d}\nu$

$$= \frac{4\pi\nu^2\,\mathrm{d}\nu}{c^3}$$

Directly applicable to

- Planck's Radiation Law
- Debye's theory of specific heats of solids
- Fermi energy level (Problem 9.19)