Chapter - 4 "Deflection of Beams"

Aim :- ① Aim of this chapter is to drive expression For maximum slope & maximum deflection for varies beams under different loading Conditions.

② The maximum deflection and maximum slope expression are used in the design of beams and design of shaft based on rigidity Criterion and in the determination of natural frequences of shaft under transverse vibration

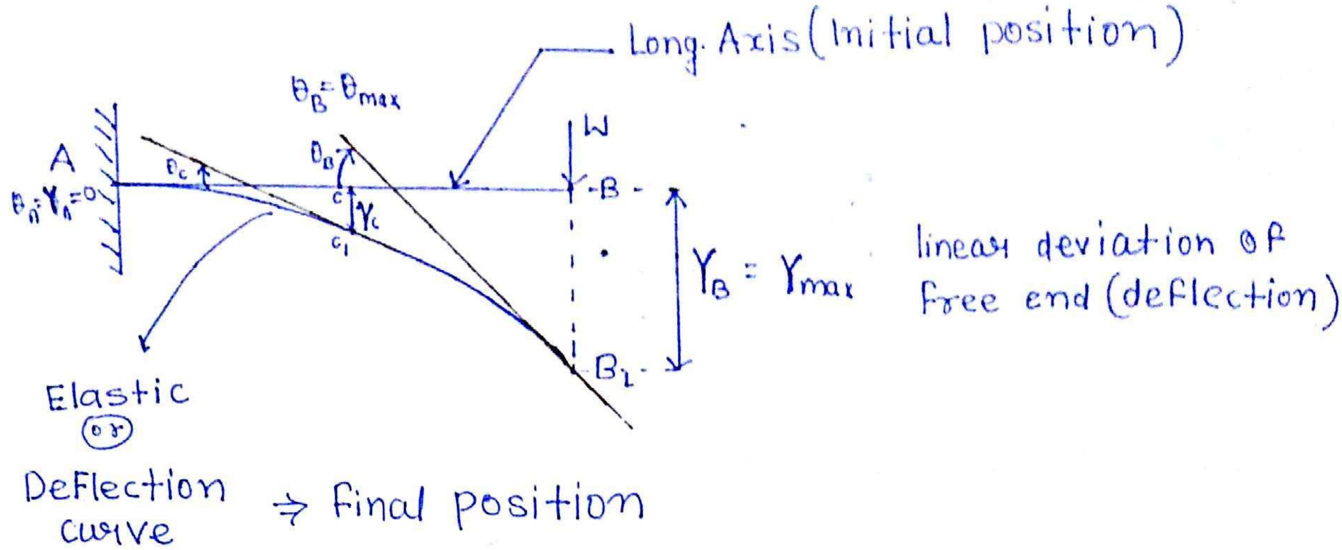
③ Deviation of axis of structural member under TSL is known as bending and deviation of axis of structral member under ACL is known as buckling.

④ Beam undergoes two deviation (linear and angular) while it is bending.

⑤ Linear deviation of Longitudnal axis of beam is know as deflection.

⑥ Angular deviation of L.A. of beam is known as slope

⑦ Slope and deflection due to shear force are neglected because they are smaller in magnitude in Comparision to slope & deflection due to bending moment.



$\Rightarrow$ Final position

Elastic Curve — (i) Circular arc [$\because$ S.F. = 0 & B.M. = Const.]

(ii) Straight line [S.F. = 0 & B.M. = 0]

(iii) Parabola [S.F. $\neq$ 0 & B.M. = Variable]

* θ - Always measure in direction of Bending moment

$$(\text{Slope})_B = \tan\theta_B = \left(\frac{dY}{dx}\right)_{AB} = \frac{Y_B - Y_A}{X_B - X_A}$$

$$(\text{Slope})_B = (\theta)_B = \left(\frac{dY}{dx}\right)_{AB} \Rightarrow \tan\theta_B \simeq \theta_B$$

$$(\theta)_C = \left(\frac{dY}{dx}\right)_{AC} = \frac{Y_C - Y_A}{X_C - X_A}$$

Sign Convention

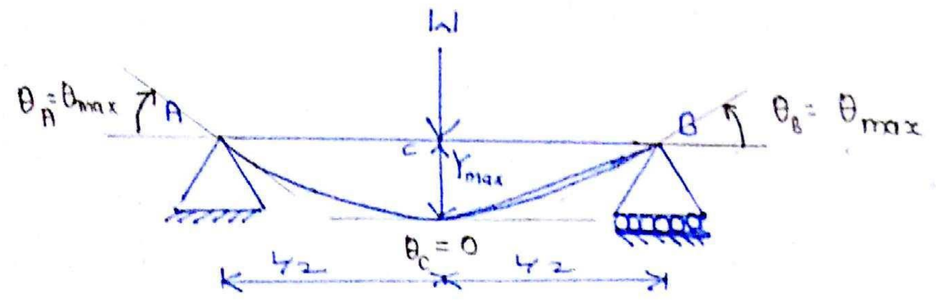
$\theta_{ACW} = +Ve$ $\qquad$ $Y_{upward} = +Ve$

$\theta_{CW} = -Ve$ $\qquad$ $Y_{downward} = -Ve$

In cantilever beam

At fixed end: $\theta = 0$ & $Y = 0$

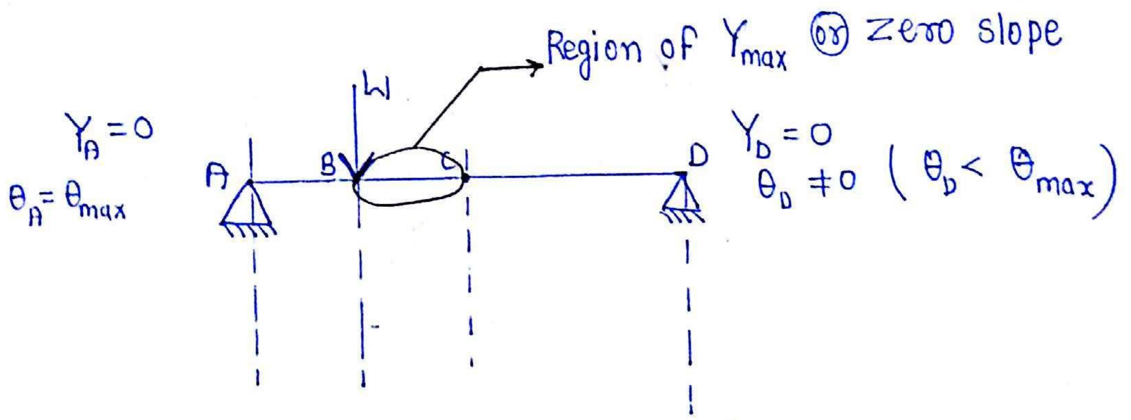
At free end: $\theta = \theta_{max}$ & $Y = Y_{max}$



In symmetrically loaded S.S.B.

* At mid span, $\theta = 0$ & $Y = Y_{max}$
* At S.S. ends, $Y = 0$ & $\theta = max$.
* slope of at the ends are equal and unlike

* Under any loading Conditions In S.S.B., slope is zero at X-S/c where deflection is maximum.



In Unsymmetrically loaded S.S.B.

(a)* Deflection is maximum in the region between point of application of load & mid span.

* Slope at the support are unequal & Unlike.

To determine Y_{max}.

1. $\theta_{x-x} = \underline{\qquad} = 0$

$\Rightarrow x = x^*$

(or) $\frac{d(Y_{x-x})}{dx} = 0$

2. $Y_{max} = (Y_{x-x})_{x=x^*} =$

* θ_{max} nearby the applied load.

Double Integration Method :-

$$\boxed{\frac{1}{R_{x-x}} = \frac{d^2 Y_{x-x}}{dx^2}}$$

where R_{x-x} = Radius of Curvature of Elastic Curve at x-x

$$R_{x-x} = \left(\frac{E\, I_{N.A.}}{M}\right)_{x-x}$$

$$\boxed{M_{x-x} = (E\, I_{N.A.})_{x-x} \frac{d^2 Y_{x-x}}{dx^2}}$$

$\Rightarrow$ Governing diff. equation in term of B.M. slope & deflection at x-s/c x-x

M_{x-x} = B.M. at x-s/c x-x

Y_{x-x} = defn at x-s/c. x-x

$\frac{dY_{x-x}}{dx}$ = slope at x-s/c x-x

$$\frac{dY_{x-x}}{dx} = \tan\theta$$

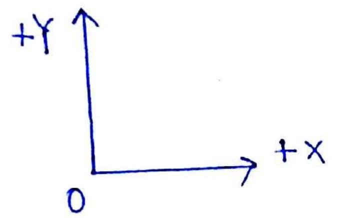
For small angular derivations

$$\tan\theta \simeq \theta$$

Hence, Slope at X-S/c X-X $\quad \frac{dY_{x-x}}{dx} = \theta$

$(EI)_{X-X}$ = Flexure Rigidity of X-S/c X-X

Sign Convention :-



(a) Bending moment

Sagging bending = +Ve

Hogging bending = -Ve

(b) Deflection

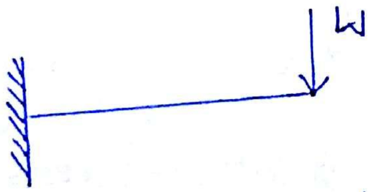
Y_{upward} = +Ve

$Y_{downward}$ = -Ve

(c) Slope (θ)

θ_{ACW} = +Ve

θ_{CW} = -Ve



Steps :- 1. $(BM)_{X-X} = M_{X-X}$

2. $(EI_{N.A.})_{X-X}\left[\frac{d^2Y_{X-X}}{dx_{X-X}^2}\right] = M_{X-X}$

3. $EI_{N.A.}\left[\frac{dY_{x-x}}{dx_{x-x}} \text{ (or) } \theta_{x-x}\right] = \int M_{x-x} + C_1$

$\Downarrow$

det. by using B.C. Condr for slope

4. $(EI_{N.A})[Y_{x-x}] = \iint M_{x-x} + C_1 x + C_2$

$\Downarrow$

Det by using B.C. of deflection

In presence of Concentrated Moments (M)

$$\theta_{max} = \frac{1}{C_1}\left[\frac{ML}{EI_{N.A.}}\right] \; ; \; Y_{max} = \frac{1}{C_2}\left[\frac{ML^2}{EI_{N.A.}}\right]$$

In presence of Concentrated point load (W)

$$\theta_{max} = \frac{1}{C_1}\left[\frac{WL^2}{EI_{N.A.}}\right] \; ; \; Y_{max} = \frac{1}{C_2}\left[\frac{WL^3}{EI_{N.A.}}\right]$$

In presence of D.L. (w)

$$\theta_{max} = \frac{1}{C_1}\left[\frac{wL^3}{EI_{N.A.}}\right] \; ; \; Y_{max} = \frac{1}{C_2}\left[\frac{wL^4}{EI_{N.A.}}\right]$$

In Case of UDL, $w = \frac{W}{L}$

In case of UVL, $w = \frac{2W}{L}$

$$\boxed{\frac{Y_{max}}{\theta_{max}} = \left[\frac{C_1}{C_2}\right] L}$$

* Deflection is always one order higher than slope

* $\left[\theta_{max} \;\&\; Y_{max}\right] \propto \left(\frac{1}{EI_{N.A}}\right)$ Inversaly proportion to Flexural rigidity (EI)

Stiffness

* $k = \dfrac{W}{Y_{max}} = C_2\left[\dfrac{EI_{N.A.}}{L^3}\right]$

* Stiffness $\boxed{k \propto EI_{N.A.}}$

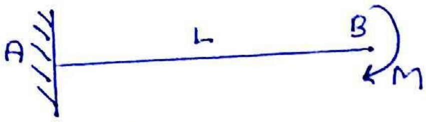
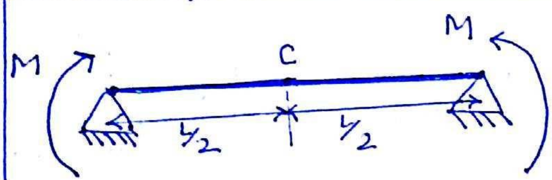
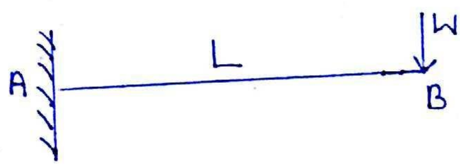
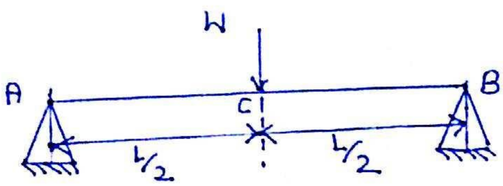
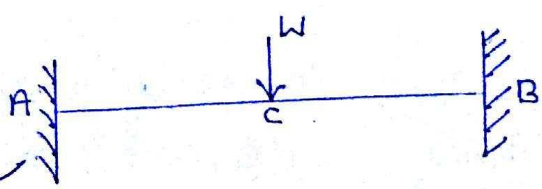
$k_{axial} = \dfrac{AE}{L}$

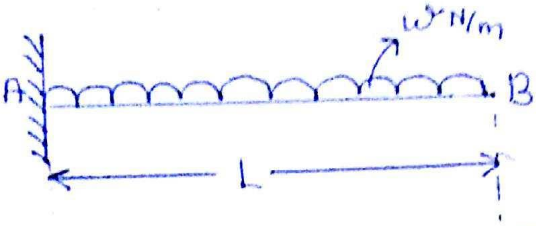
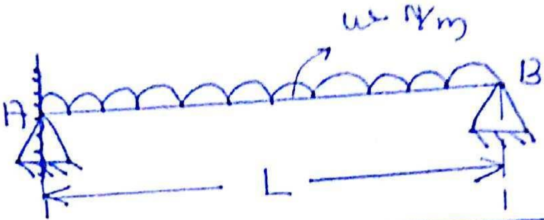
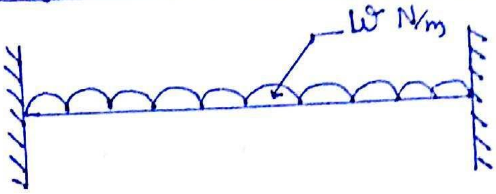
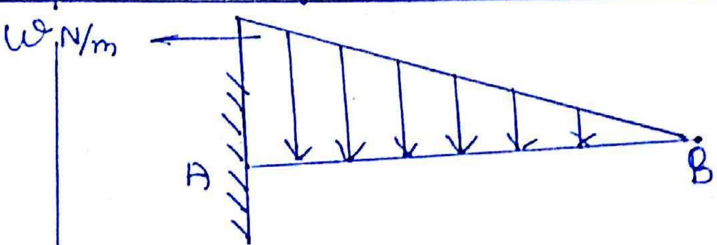
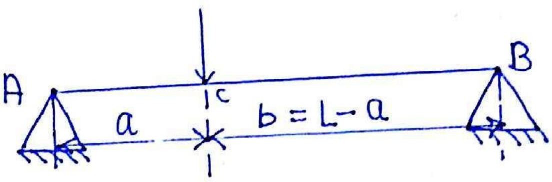
$k_{torsional} = \dfrac{GJ}{L}$

* $EI_{N.A.}(\uparrow) \Rightarrow k_{beam}(\uparrow)$

$\Rightarrow Y_{max}(\downarrow)$

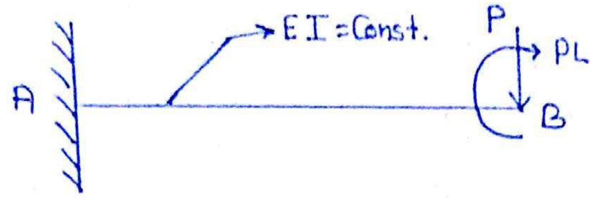
$\Rightarrow \theta_{max}(\downarrow)$

S. No.	($EI_{N.A.}$ = Const.) Type of Beam	For θ_{max} C_1	For Y_{max} C_2
1.		1	2
2.		2	8
3.		2	3
4.		16	48
5.		—	192

6.		6	8
7.		24	$\frac{384}{5}$
8.		—	384
9.		24	30
10.		$\theta_C = \frac{Wb}{3EI_{N.A.}L}\left[a^2 - ab\right]$ $Y_C = \frac{Wa^2b^2}{3EI_{N.A.}L}$	

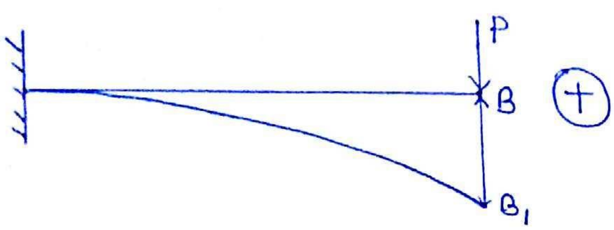
	M		W		UDL		UVL	
C.V.	1	2	2	3	6	8	24	30
SSB	2	8	16	48	24	$\frac{384}{5}$		
			−	192	−	384		

Question

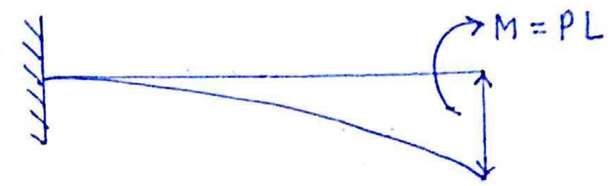


Using principle of super position

(1)



(2)



$$\theta_1 = \frac{1}{c_1}\left[\frac{PL^2}{EI}\right] = \frac{1}{2}\left[\frac{PL^2}{EI}\right] (\circlearrowright)$$

$$Y_1 = \frac{1}{C_2}\left[\frac{PL^3}{EI}\right] = \frac{1}{3}\left[\frac{PL^3}{EI}\right] (\downarrow)$$

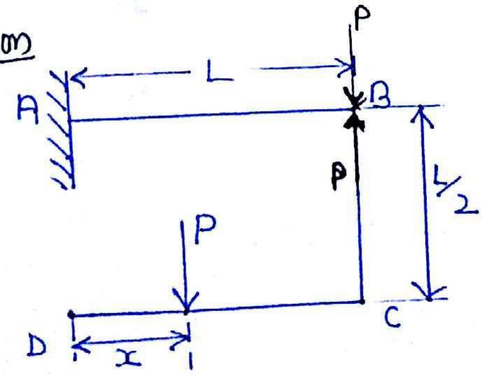
$$\theta_2 = \left[\frac{1}{C_1}\right]\left[\frac{ML}{EI}\right] = \frac{1}{1}\left[\frac{(PL)\,L}{EI}\right] (\circlearrowright)$$

$$Y_2 = \frac{1}{c_2}\left[\frac{ML^2}{EI}\right] = \frac{1}{2}\left[\frac{PL^3}{EI}\right] (\downarrow)$$

$$\theta_B = \theta_1 + \theta_2 = \frac{3}{2}\,\frac{PL^2}{EI} \quad (\circlearrowright)$$

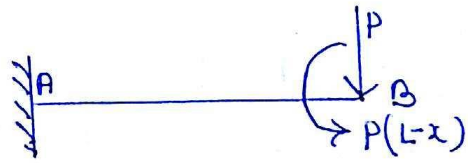
$$Y_B = Y_1 + Y_2 = \frac{5}{6}\,\frac{PL^3}{EI} \quad (\downarrow)$$

Question



Determine the value of 'x' if $Y_B = 0$

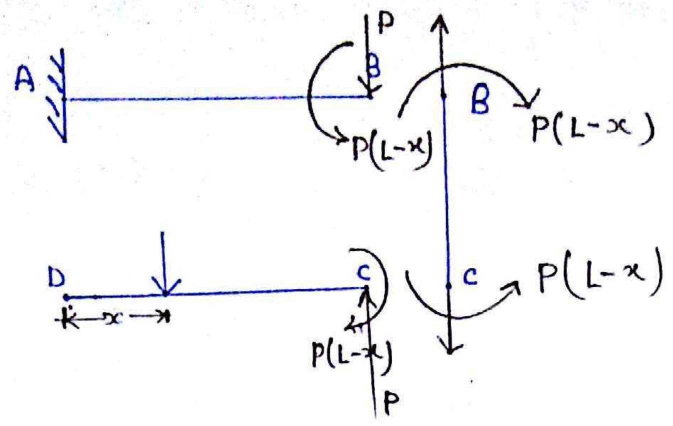
$\equiv$



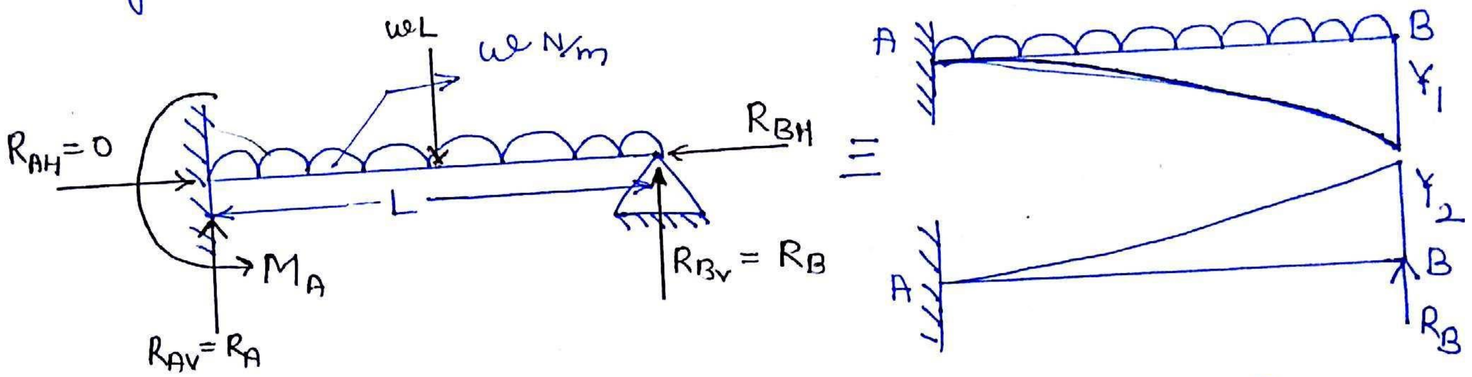
$$Y_B = Y_1 + Y_2 = \left(-\frac{1}{3}\right)\frac{PL^3}{EI} + \frac{ML^2}{2EI} = 0$$

$$\frac{-PL^3}{3EI} + \frac{P(L-x)L^2}{2EI} = 0$$

$$\frac{2}{3}L = L - x \Rightarrow \boxed{x = L/3}$$



Question For the proporpped Cantilever beam as shown in Fig determine the support reaction.



$$\Sigma V = 0 \Rightarrow R_A + R_B = wL \quad — (1)$$

$$\Sigma M_A = 0 \Rightarrow -M_A + wL\left(\frac{L}{2}\right) - R_B(L) \quad — (2)$$

$$Y_B = Y_1 + Y_2 = 0$$

$$Y_B = -\left(\frac{1}{8}\right)\frac{wL^4}{EI} + \frac{1}{3}\left(\frac{R_B L^3}{EI}\right) = 0$$

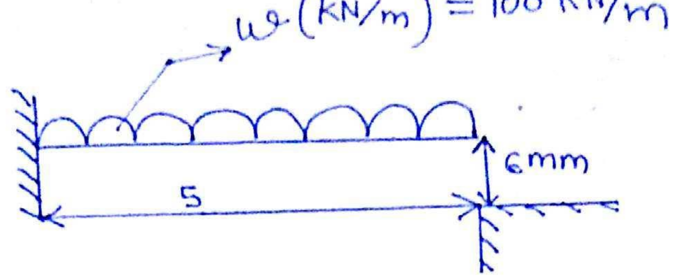
$$R_B = \frac{3}{8}(wL) \uparrow$$

From eqⁿ (1), $R_A = \frac{5}{8} wL \uparrow$.

From eqⁿ (2) $M_A = \frac{wL^2}{8}$

Question: 12

workbook



No use of Extra data

Assuming $R_B = 0$ $[\because Y_B \leq 6 \text{ mm}]$

Cantilever beam

$$Y_B = \frac{wL^4}{8EI_{N.A.}}$$

$$Y_B = \frac{100 \times (5)^4}{8 \times 78125 0 \times} = 10 \text{mm}$$

$R_B \neq 0$ $[\because Y_B > 6 \text{ mm}]$ & St. Indt. beam

$$Y_B = Y_1 + Y_B = 0$$

$$-\frac{wL^4}{8EI} + \frac{R_B L^3}{3EI} = -6 \text{ mm}$$

$$R_B = 75 \text{ KN}$$

$$\Sigma V = R_A + R_B = 500 \text{ kN}$$

$$R_A = 500 - 75$$

$$R_A = 425 \text{ kN } (\uparrow)$$

Question For the beam as shown in Fig determine

(i) spring reaction

(ii) Deflection of spring.

Data: -

$W = 20\,kN$

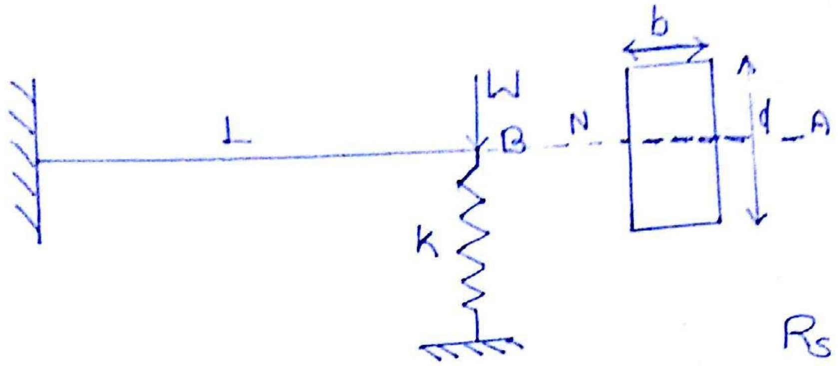
$k = 500\ N/mm$

$E = 200\ GPa$

$L = 2\ m$

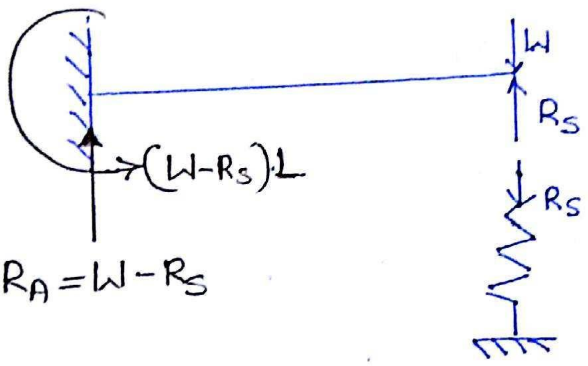
$b = 50\,mm$

$d = 100\ mm$



$$R_s = kx = k\delta_s$$

Sol^n



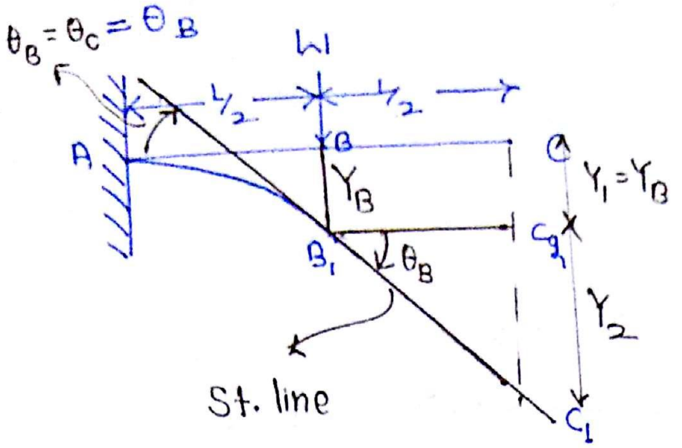
$$Y_B = Y_1 + Y_2 = -\delta_s$$

$$\frac{-WL^3}{3EI} + \frac{R_s L^3}{3EI} = -\frac{R_s}{K}$$

$$R_s = \frac{KWL^3}{KL^3 + 3E\,I_{N.A.}}$$

$$\delta_s = \frac{WL^3}{KL^3 + 3E\,I_{N.A.}}$$

Question Find deflection at C



$\Delta B_1 C_2 C_1$

$$\tan\theta_B = \frac{Y_2}{L/2}$$

$$Y_2 = \theta_B \frac{L}{2} \qquad \because \tan\theta_B \simeq \theta_B$$

$$\boxed{Y_C = Y_{max} = Y_1 + Y_2 = Y_B + \theta_B (L/2)}$$

$$\theta_B = \frac{1}{2}\frac{W(L/2)^2}{EI} = \frac{WL^2}{8EI}$$

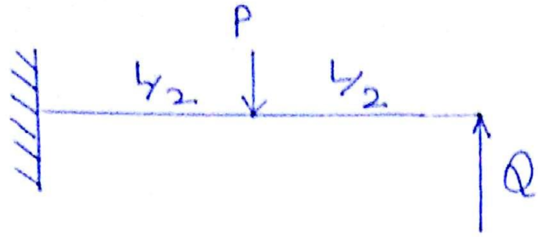
$$Y_B = \frac{1}{3}\frac{W(L/2)^3}{EI} = \frac{WL^3}{24EI}$$

$$\theta_{max} = \theta_C = \theta_B = \frac{WL^2}{8EI} (\circlearrowright)$$

$$Y_{max} = \frac{WL^3}{24EI} + \frac{WL^2}{8EI}\left(\frac{L}{2}\right)$$

$$Y_{max} = \frac{5}{48}\left[\frac{WL^3}{EI_{N.A.}}\right] (\downarrow)$$

Question

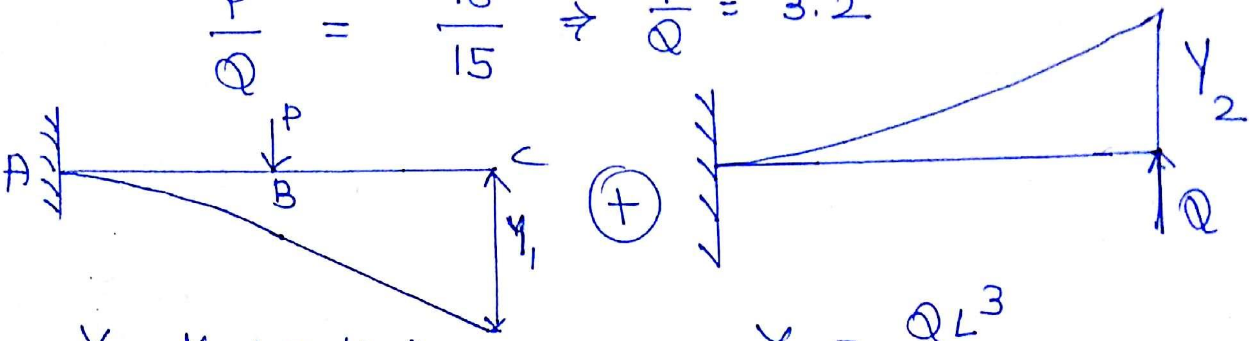


determine $\frac{P}{Q} = ?$ if $Y_c = 0$

Sol^n

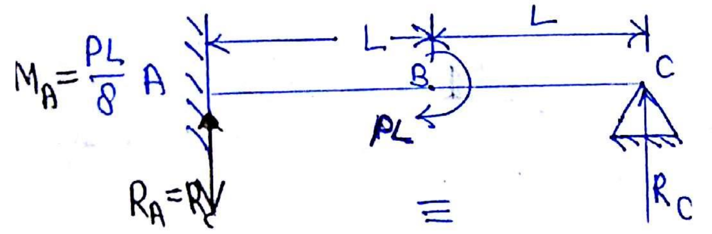
$$\frac{5}{48}\frac{PL^3}{EI} = \frac{QL^3}{3EI}$$

$$\frac{P}{Q} = \frac{48}{15} \Rightarrow \frac{P}{Q} = 3.2$$



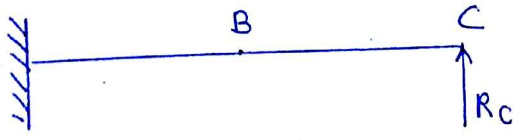
$$Y_1 = Y_B + \theta_B\left(\frac{L}{2}\right) \qquad Y_1 = \frac{QL^3}{3EI}$$

Ques

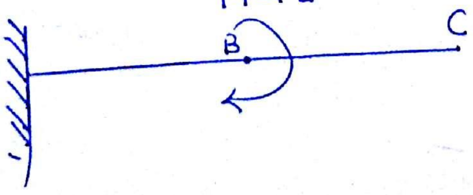


Find $R_c = ?$

=



$$Y_1 = \frac{R_c(2L)^3}{3EI} = \frac{8R_cL^3}{3EI}$$



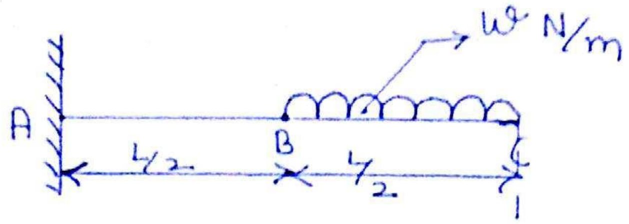
$$\boxed{Y_2 = Y_B + \theta_B L} = \frac{ML^2}{2EI} + \frac{ML}{EI}(L)$$

$$Y_2 = \frac{PL^3}{2EI} + \frac{PL^3}{EI} = \frac{3}{2}\frac{PL^3}{EI}$$

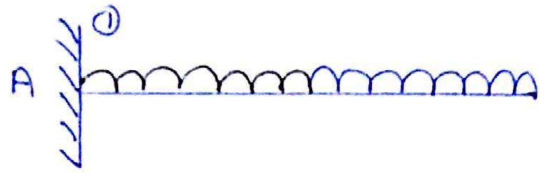
$$Y_c = Y_1 + Y_2 = 0 \Rightarrow \frac{8}{3}\frac{R_cL^3}{EI} + \left(-\frac{3}{2}\frac{PL^3}{EI}\right) \Rightarrow R_c = \frac{9}{16}P\ (\uparrow)$$

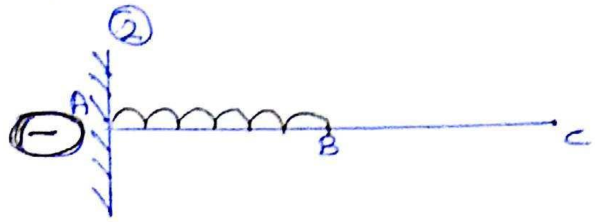
$$\Rightarrow PL - \frac{9}{16}P(L \times 2) = M_A = \frac{PL}{8} \Leftarrow \quad \Rightarrow R_A = \frac{9}{16}P\ (\downarrow)$$

Ques Find slope and deflection at C i.e. $\theta_c = ?$
$Y_c = ?$



Soln





$$\theta_1 = \frac{wL^3}{6EI} (\circlearrowleft) \; ; \; Y_1 = \frac{wL^4}{8EI} (\downarrow) \qquad \theta_2 = \frac{w(L/2)^3}{6EI} = \frac{wL^3}{48EI} (\circlearrowleft)$$

$$Y_1 = \frac{wL^4}{8EI} (\downarrow)$$

$$\boxed{Y_2 = Y_B + \theta_B (L/2)}$$

$$Y_2 = \frac{w(L/2)^2}{8EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right)$$

$$Y_2 = \frac{7}{384} \frac{wL^4}{EI}$$

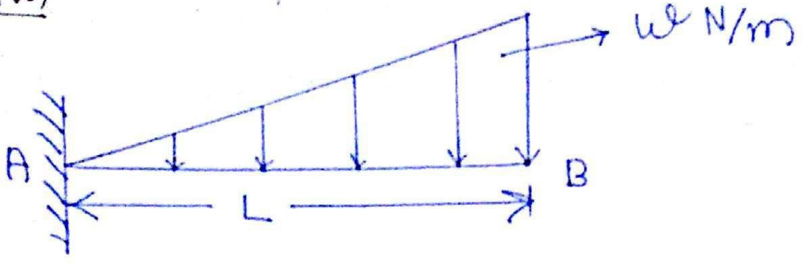
$$Y_c = Y_1 - Y_2$$

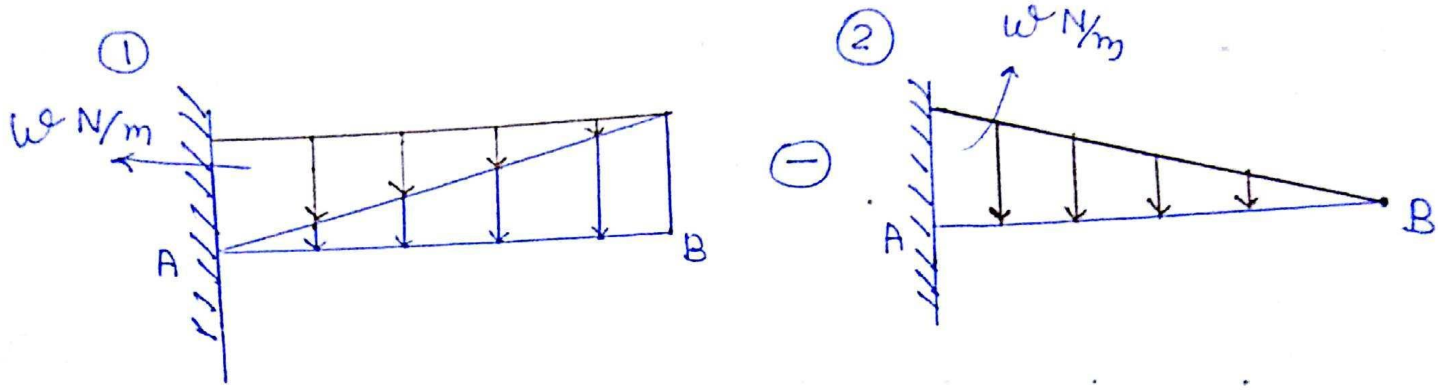
$$Y_c = \frac{wL^4}{EI}\left(\frac{1}{8} - \frac{7}{384}\right) = \left(\frac{41}{384}\right)\left(\frac{wL^4}{EI}\right)(\downarrow)$$

$$\theta_c = \theta_1 - \theta_2$$

$$\theta_c = \frac{wL^3}{EI}\left(\frac{1}{6} - \frac{1}{48}\right) = \frac{7}{48} \frac{wL^3}{EI} (\circlearrowleft)$$

Question





$$(Y_B)_{I} = \frac{wL^4}{8EI} (\downarrow) \qquad (Y_B)_{II} = \frac{wL^4}{30EI} (\downarrow)$$

$$Y_B = (Y_B)_1 - (Y_B)_{II}$$

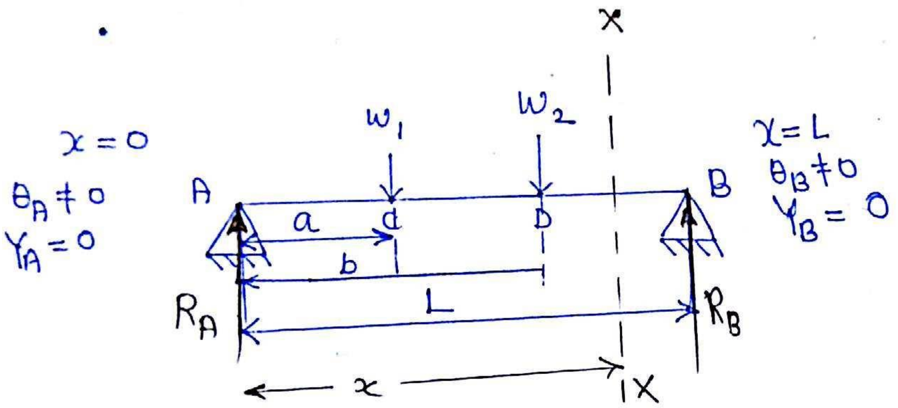
$$Y_B = \frac{wL^4}{EI}\left(\frac{1}{8} - \frac{1}{30}\right) = \frac{11}{120}\frac{wL^4}{EI}$$

$$\theta_B = \frac{wL^3}{6EI} - \frac{wL^3}{24EI} = \frac{wL^3}{8EI}$$

Macauly's Method :–

(Modifield double Integration method)

$$\boxed{M_{x-x} = (E I_{N.A.})_{x-x} \frac{d^2 Y_{x-x}}{d x^2}}$$



⊛ Consider a X-S/c in last segment and write BM

$$M_{x-x} = R_A(x) - W_1(x-a) - W_2(x-b)$$

⊛ For any x value if term is -ve then discard it

e.g. For $x = a/2$

$M_{x-x} = R_A(a/2) \quad \because (a/2 < a < b)$

Neglect two

Now

$$E I_{N.A.} \frac{d^2 y_{x-x}}{d x^2} = R_A x - W_1(x-a) - W_2(x-b)$$

$$E I_{N.A.} \left[\frac{dy}{dx^2} \text{ (or) } \theta_{x-x} \right] = R_A \frac{x^2}{2} - \frac{W_1(x-a)^2}{2} - \frac{W_2(x-b)^2}{2} + C_1$$

in machulyl's method

$$\int (x-a)\,dx = \frac{x^2}{2} - ax \quad \times$$

$$\int (x-a)^2\,dx = \frac{(x-a)^2}{2} \quad \checkmark$$

$$EI_{N.A.}\left[Y_{x-x}\right] = R_A\frac{x^3}{6} - \frac{W_1(x-a)^3}{6} - \frac{W_2(x-b)^2}{6} + C_1x + C_2$$

$\Rightarrow$ $x = 0 \Rightarrow Y = 0$

$$EI_{N.A.}\left[0\right] = 0 - 0 - 0 + 0 + C_2$$

at $x = 0$
$(x-b)$ & $(x-a)$ become $-ve$ so
discard them

$\Rightarrow$ $x = L \Rightarrow Y = 0$

$$EI_{N.A}\left[0\right] = \frac{R_A L^3}{6} - \frac{W_1(L-a)^3}{6} - \frac{W_2(L-b)^3}{6} + C_1 L$$

$C_1 = B$ Const.

$\Rightarrow$ Equation for slope

$\Rightarrow$

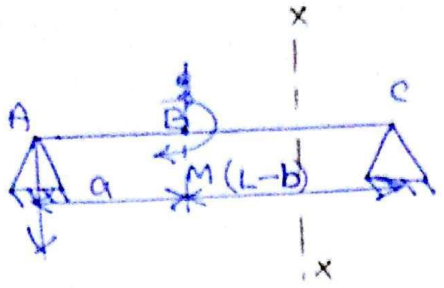
$$EI_{N.A.}\left[\frac{dY_{x-x}}{dx}\right] = \frac{R_A x^2}{2} - \frac{W_1(x-a)^2}{2} - \frac{W_2(x-b)^2}{2} + B$$

$\Rightarrow$ Equation for deflection

$$EI_{N.A.}\left[Y_{x-x}\right] = \frac{R_A x^3}{6} - \frac{W_1(x-a)^3}{6} - \frac{W_2(x-b)^3}{6} + Bx$$

if $x < a$ discard $(x-a)$ & $(x-b)$ $a < x < b$ discard $(x-b)$

⇒



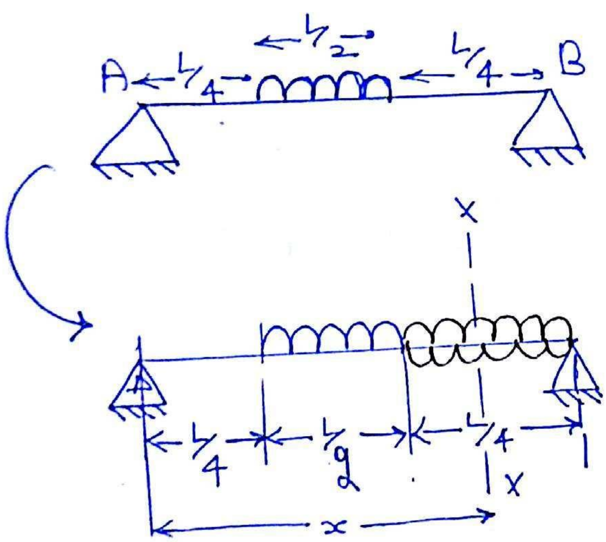
$\rightarrow M_{x-x} = R_A x - M\langle x-a \rangle^0$

$R_A = \frac{M}{L}$

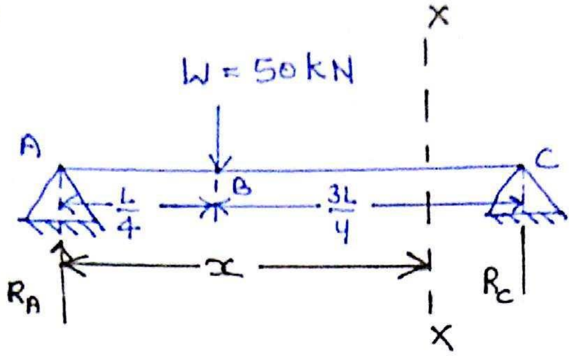
$M_{x-x} = R_A x - M$ ✗

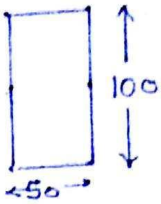
$$\boxed{M_{x-x} = R_A x - M(x-a)^0}$$ ✓

⇒



Question





Determine

(a) θ_A & θ_C

(b) Y_B & Y_{max}

if $L = 2m$; $E = 200\,GPa$

$$M_{x-x} = \frac{3}{4}W(x) - W\left(x - \frac{L}{4}\right) \qquad R_A = \frac{W \times \frac{3L}{4}}{L} = \frac{3W}{4}$$

$$EI\frac{d^2y}{dx^2} = \frac{3}{4}Wx - W\left(x - \frac{L}{4}\right)$$

$$EI\frac{dy}{dx} = \frac{3}{4}W\frac{x^2}{2} - \frac{W}{2}\left(x - \frac{L}{4}\right)^2 + C_1 \quad -(1)$$

$$EI[Y] = \frac{3}{24}Wx^3 - \frac{W}{6}\left(x - \frac{L}{4}\right)^3 + C_1x + C_2 \quad -(2)$$

$x = 0$, $Y = 0$ $\quad 0 = 0 - 0 + 0 + C_2$ $\quad C_2 = 0$

$x = L$, $Y = 0$ $\Rightarrow 0 - \frac{1}{8}WL^3 - \frac{27WL^3}{64} + C_1L =$

$$C_1 = WL^2\left(\frac{1}{8} + \frac{27}{64}\right)$$

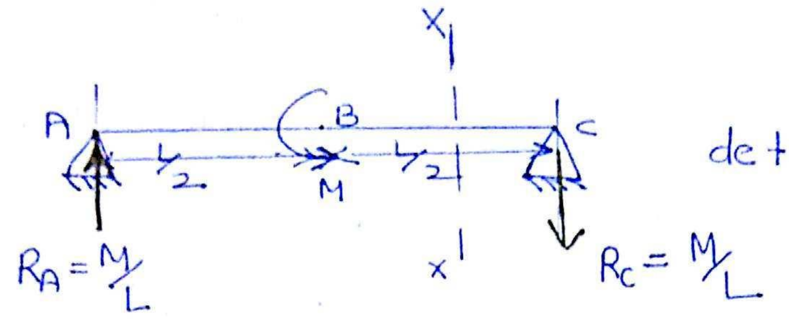
$$WL^2\left(\frac{8+27}{64}\right)$$

$$C_1 = \frac{35}{64}WL^2$$

$$Y = \frac{1}{EI}\left[\frac{3Wx^3}{24} - \frac{W}{6}\left(x - \frac{L}{4}\right)^3 + \frac{35}{64}WL^2x\right]$$

$$\theta = \frac{1}{EI}\left[\frac{3Wx^2}{8} - \frac{W}{2}\left(x - \frac{L}{4}\right)^2 + \frac{35}{64}WL^2\right]$$

Question



det θ_B & Y_B

$$M_{x-x} = \frac{M}{L}x - M(x - L/2)^0$$

$$EI\frac{dy}{dx} = \frac{Mx^2}{2L} - M(x - L/2) + C_1$$

$$EI[y] = \frac{Mx^3}{6L} - \frac{M}{2}(x - L/2)^2 + C_1 x + C_2$$

$x = 0$, $y = 0$ $\Rightarrow C_2 = 0$ $\quad$ $x = L$, $y = 0$ $\Rightarrow 0 = \frac{ML^3}{6L} - \frac{M}{2}(L/2)^2 + C_1 L = 0$

$$C_1 = ML^2\left(\frac{1}{8} - \frac{1}{6}\right)$$

$$C_1 = -\frac{ML^2}{24}$$

$$\theta = \frac{dy}{dx} = \frac{1}{EI}\left[\frac{Mx^2}{2L} - M(x - L/2) - \frac{ML}{24}\right]$$

$$\theta_B\Big|_{x=L/2} = \frac{1}{EI}\left[\frac{M}{2L}\frac{L^2}{4} - \frac{ML}{24}\right] = \frac{ML}{EI}\left(\frac{1}{8} - \frac{1}{24}\right) = \frac{ML}{EI}\,\frac{3-1}{24}$$

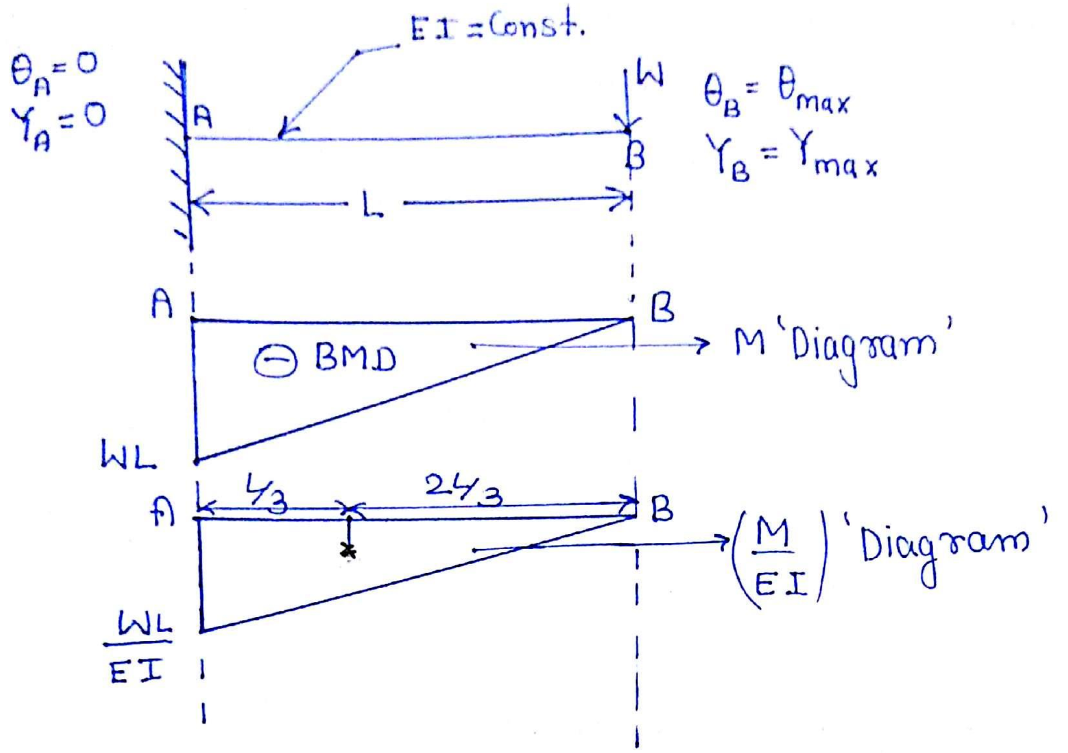
$$\theta_B = -\frac{ML}{12EI} \text{ or } \frac{ML}{12EI}\ (cw)$$

$$Ey = \frac{1}{EI}\left[\frac{Mx^3}{6L} - \frac{M}{2}(x - L/2)^2 + \frac{ML}{24}x\right]$$

$$y\Big|_{x=\frac{L}{2}} = \frac{ML^2}{EI}\left[\frac{1}{48} - \right.$$

Moment Area Method :-

(i) Draw B.M.D.



Mohr's Theorems

First theorem :- Diffrence of two slope between two x-S/c (one is zero slope and other one is non zero slop) is equal to area of M/EI diagram

$$\theta_B - \theta_A = \text{Area of } \frac{M}{EI} \text{ diagram between B \& A}$$

$$\theta_{max} - 0 = \frac{1}{2} \times L \times \left(\frac{-WL}{EI}\right)$$

$$\theta_{max} = \frac{-WL^2}{2EI} \text{ or } \frac{WL^2}{2EI} \ (\circlearrowright) \text{ Clockwise}$$

Second theorem :- Diff. of deflection between two x-S/c
(simplified) moment of area of $\frac{M}{EI}$ Diagram.

↳ is valid when point of zero slope is know.

$Y_B - Y_A$ = moment of Area of $\frac{M}{EI}$ diag. betn B & A

(or) $Y_B - Y_A = A\bar{x}$ of $\frac{M}{EI}$ diag. betn B & A

$$Y_{max} - 0 = \frac{-WL^2}{2EI} \times \frac{2L}{3}$$

$\bar{x}$ = from non zero slope

$$Y_{max} = \frac{-WL^3}{3EI} \text{ (or) } \frac{WL^3}{3EI} (\downarrow)$$

if $EI_{N.A.}$ = constant

I theorem $\Rightarrow$ $\theta_B - \theta_A = \frac{1}{EI_{N.A.}}$ [Area of BMD betn B & A]

II theorem $\Rightarrow$ $Y_B - Y_A = \frac{1}{EI_{N.A.}}$ [$A\bar{x}$ of BMD betn B & A]

Actual II theorem: $\Rightarrow$ when point of zero slope unknown.

$$(Y_B - Y_A) + (x_A\theta_A - x_B\theta_B) = A\bar{x} \text{ of } \frac{M}{EI} \text{ dia betn B \& A}$$

in previous example $x_A = L$; $\theta_A = 0$

$x_B = 0$; $\theta_B \neq 0$

So $Y_B - Y_A = A\bar{x}$ of $\frac{M}{EI}$ dia. betn B & A

Conclusion :-

(i) Simplifield second theorem should be use when point of zero slope is known (for all cantilever beam and SSB under symmetric loading conditions.

(ii) when simplifield II theorem is use two X-S/c should be selected in such a way that one should be ~~because~~ a zero slope X-S/c and another one is non-zero slope X-S/c.

(iii) Non zero X-S/c should be selected in such a way that it should be X-S/c where slope and deflection are to be determine.

(iv) Non zero slope X-S/c should be considered as origion and zero slope X-S/c is considered as refrence X-S/c

(v) $\bar{X}$ should be measure from origion (i.e. non zero slope X-S/c)

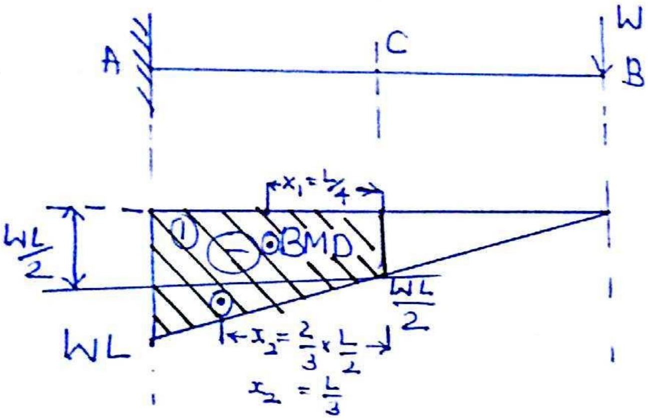
(vi) Area of BMD should be use for the calculation of slope and deflection of beam whose flexure rigidity is constant

(vii) Area of M/EI diagram should be use for the calculation of slope & deflection of beam whose flexure rigidity is variable.

(viii) Actual second theorem should be use when zero slope X-S/c is unknown (i.e. for SSB under unsymmetric loading)

Quest

Determine slope & deflection at a x-S/c located at a distance $\frac{L}{2}$ from fixed of a Cantilever beam when it is loaded by concentrated point load at the free end in the downward dir^n



Assum EI Const.

$\theta_A = 0$

$Y_A = 0$

Ref. x-S/c → Zero slope x-S/c (A)

Origin → Non Zero slope x-S/c (C)

$A_1 = -\frac{WL}{2} \times \frac{L}{2}$

$A_2 = -\frac{1}{2} \times \frac{WL}{2} \times \frac{L}{2}$

$$\theta_C - \theta_A = -\frac{1}{2} \times \left(\frac{L}{2}\right) \times \left(\frac{3}{2}\frac{WL}{EI}\right)$$

$$\theta_C = -\frac{3}{8}WL^2 \text{ or } \frac{3}{8}\frac{WL^2}{EI} \circlearrowright$$

or $\theta_C - 0 = \frac{1}{EI}(A_1 + A_2)$

$$= \frac{-WL^2}{EI}\left[\frac{1}{4} + \frac{1}{8}\right]$$

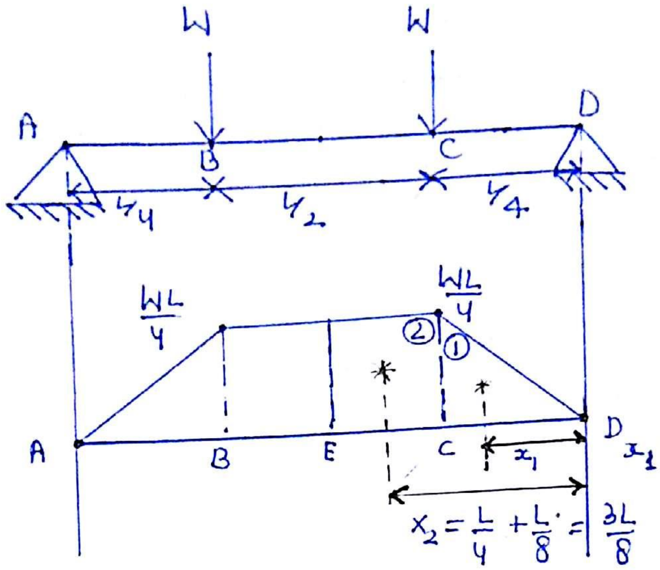
$$\theta_C = \frac{3}{8}WL^2 \circlearrowright$$

$$Y_C - Y_A = \frac{1}{EI_{N.A.}}\left[A_1\bar{x}_1 + A_2\bar{x}_2\right]$$

$$Y_C = \frac{1}{EI}\left[\left(\frac{-WL^2}{4} \times \frac{L}{4}\right) + \left(\frac{-WL^2}{8} \times \frac{L}{3}\right)\right]$$

$$Y_C = \frac{-5}{48}\frac{WL^3}{EI_{N.A.}} \text{ or } \frac{5WL^3}{48EI_{N.A.}} (\downarrow)$$

Prob. For SSB as shown in fig determine slope at the support and max. deflection.



$\theta_D = \theta_{max}$

$Y_D = 0$

$\bar{x}_1 = \frac{2}{3} \times \frac{L}{4} = \frac{L}{6}$

Ref X-S/c → zero slope X-S/c (E)

origion → non zero slope. X-S/c (D)

$$\theta_D - \theta_E = \frac{1}{EI}(A_1 + A_2)$$

$$= \frac{1}{EI}\left(\frac{1}{2} \times \frac{WL}{4} \times \frac{L}{4} + \left(\frac{WL}{4} \times \frac{L}{4}\right)\right)$$

$$\theta_D = \frac{3\,WL^2}{32\,EI} \;(\circlearrowright)$$

$$\theta_A = -\theta_D = \frac{3}{32}\frac{WL^2}{EI} \;(\circlearrowleft)$$

$$Y_D - Y_E = \frac{1}{EI}(A_1\bar{x}_1 + A_2\bar{x}_2) = \frac{1}{EI}\left[\frac{WL^2}{32} \times \frac{L}{6} + \frac{WL^2}{16} \times \frac{3L}{8}\right]$$

$$Y_{max} = \frac{11}{384}\left(\frac{WL^3}{EI}\right) (\downarrow)$$

Find deflection at C $Y_c = ?$

origion → C
ref. → E

$$Y_C - Y_E = \frac{1}{EI}(A\bar{x})$$

Consider Area b/w C & E

$$A = \frac{WL}{4} \times \frac{L}{4} = \frac{WL^2}{16}$$

$$\bar{x} = L/8 \text{ (from c)}$$

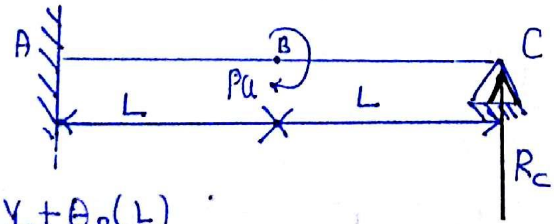
$$Y_C - \left(\frac{-11}{384}\frac{WL^3}{EI}\right) = \frac{1}{EI}\left(\frac{WL^2}{16} \times \frac{L}{8}\right)$$

$$Y_c = \frac{WL^3}{EI}\left(\frac{1}{128} - \frac{11}{384}\right)$$

$$Y_c = \frac{WL^3}{EI}\left(\frac{3 - 11}{384}\right)$$

$$Y_c = +\frac{WL^3}{EI}\left(\frac{-8}{384}\right) = \frac{8}{384}\left(\frac{WL^3}{384}\right)(\downarrow)$$

__Ques__ For the properped Contilever beam as shown in Fig determine slope at B $(\theta_B = ?)$

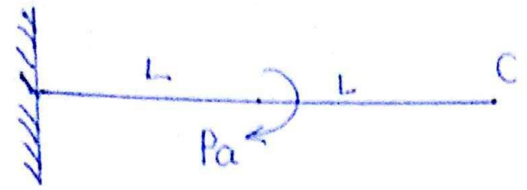


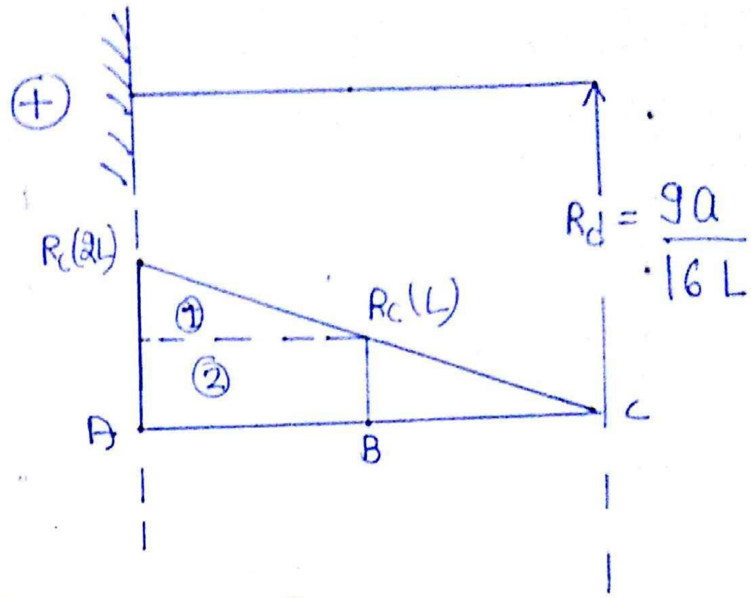
Use principle of Super position
def^n at C $Y_c = Y_{c_1} + Y_{c_2} = 0$

$$(Y_c)_1 = Y_B + \theta_B(L)$$

$$\frac{PaL^2}{2EI} + \frac{PaL}{EI}(L) = \frac{R_c(2L)^3}{3EI}$$

$$\frac{3PaL^2}{2EI} = \frac{8R_cL^3}{3EI} \Rightarrow R_c = \frac{9Pa}{16L}$$





$$\theta_B = \frac{ML}{EI}$$

$$(\theta_B)_1 = \frac{PaL}{EI}$$

Origin → B
Ref. → A

$$(\theta_B)_2 - (\theta_A)_1 = \frac{1}{EI}\left(\left(\frac{1}{2} R_c L \times L\right) + (R_c L) L\right]$$

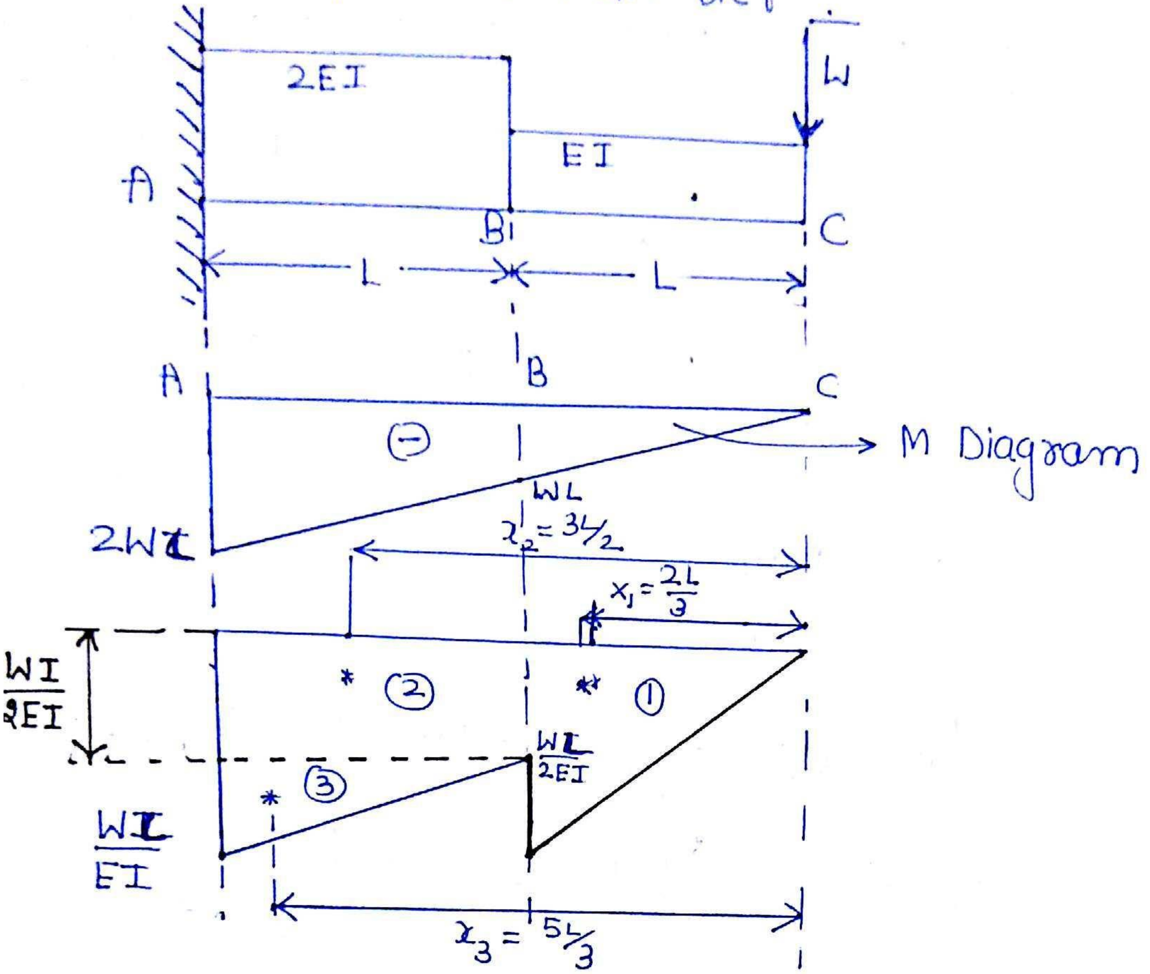
$$(\theta_B)_2 - 0 = \frac{3}{2}\frac{R_c L^2}{EI} = \frac{27 PaL}{32 EI}$$

$$\theta_B = (\theta_B)_1 + (\theta_B)_2$$

$$\theta_B = \frac{-PaL}{EI} + \frac{27 PaL}{32 EI}$$

$$\theta_B = \frac{5 PaL}{32 EI}$$

Ques. For the centilever beam as shown in fig determine max. slope and max. defn



origin → C Ref:- A

$$\theta_C - \theta_A = [A_1 + A_2 + A_3]$$

$$\theta_{max} = \left[\left(-\frac{1}{2}\frac{WL}{2EI} \times L\right) + \left(-\frac{WL}{2EI} \times L\right) + \left(-\frac{1}{2} \times \frac{WL}{2EI} \times L\right)\right]$$

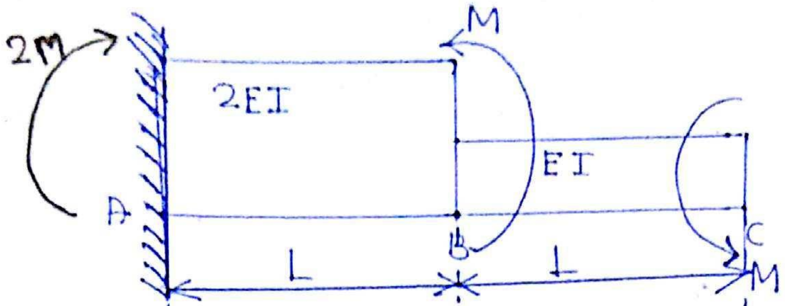
$$\theta_{max} = \frac{5}{4}\frac{WL^2}{EI} \ (\circlearrowright)$$

$$Y_C - Y_A = A_1\bar{X}_1 + A_2\bar{X}_2 + A_3\bar{X}_3$$

$$Y_{max} - 0 = \left[\left(\frac{-WL^2}{2EI} \times \frac{2L}{3}\right) + \left(\frac{-WL^2}{2EI} \times \frac{3L}{2}\right) + \left(\frac{-WL^2}{4EI} \times \frac{5L}{3}\right)\right]$$

$$Y_{max} = -\frac{3}{2}\left(\frac{WL^3}{EI}\right) \text{ or } \frac{3}{2}\frac{WL^3}{EI} \ (\downarrow)$$

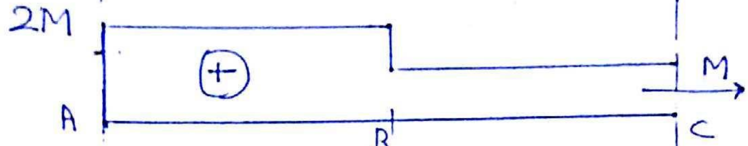
Ques.



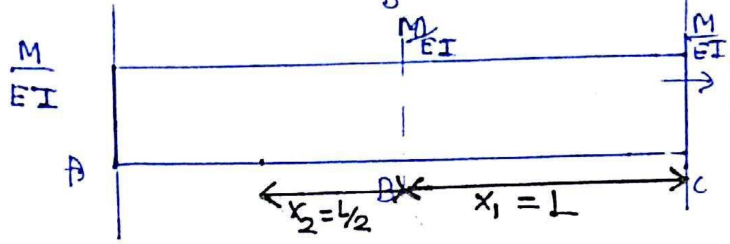
determine

(a) Y_C & $\theta_C = ?$

(b) θ_B & $Y_B = ?$



M 'diagram'



$\left(\frac{M}{EI}\right)$ diagram

Origion – C, Ref – A

$$\boxed{\begin{aligned}\theta_A &= 0 \\ Y_A &= 0\end{aligned}}$$

$$\theta_C - \theta_A = (2L)\frac{M}{EI}$$

$$\theta_C = \frac{2ML}{EI}\ (\circlearrowright)$$

$\bar{x}_1 = L$ (from C)

$$Y_C - Y_A = \frac{2ML}{EI}(L)$$

$$Y_C = \frac{2ML^2}{EI}\ (\uparrow)$$

Origion-B

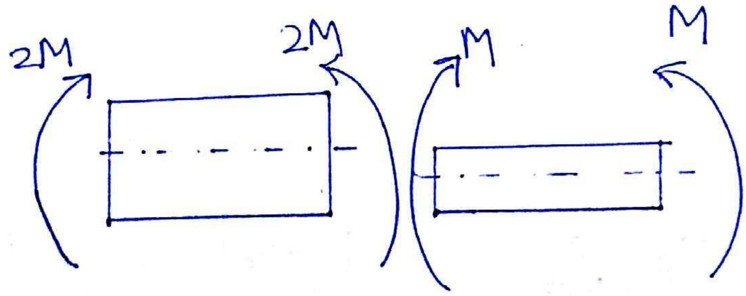
Ref-A

$$\theta_B - \theta_A = L\frac{M}{EI}$$

$$\theta_B = \frac{ML}{EI}\ (\circlearrowright)$$

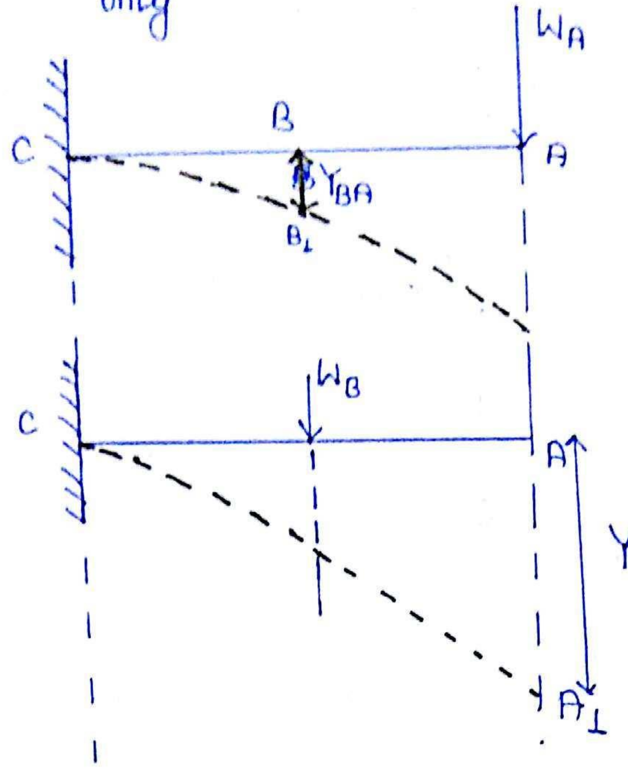
$\bar{x}_2 = L/2$ (from B)

$$Y_B = \frac{ML}{EI}\left(\frac{L}{2}\right) = \frac{ML^2}{2EI}\ (\uparrow)$$



Maxwell's Reciprocal Theorem :-

is valid only for def^n in presence concentrated point load only (Any beam)



Y_{BA} → load at A

↳ def^n at B

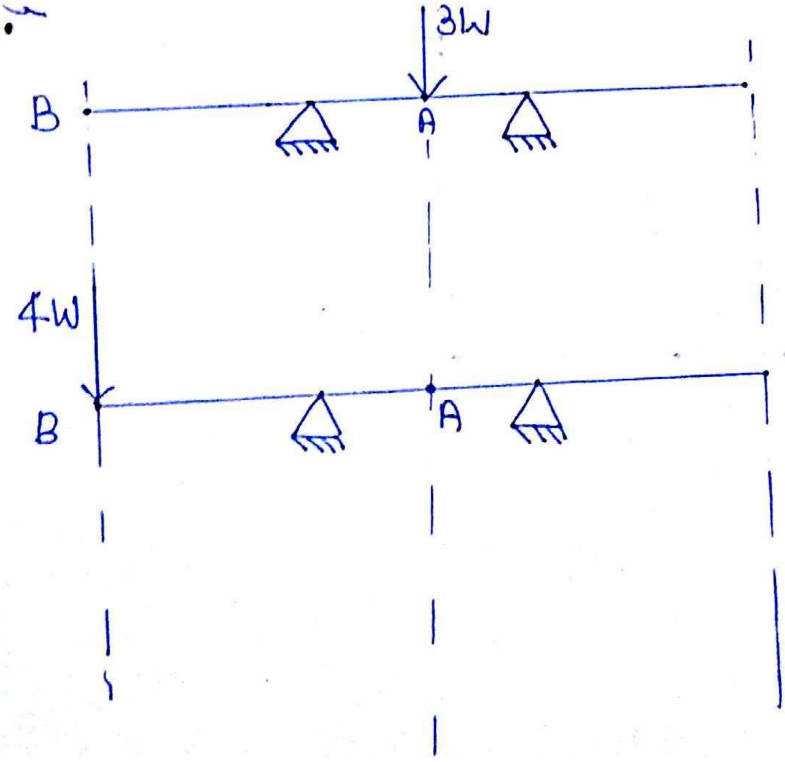
Y_{AB} → load is applied at 'B'

↳ Def^n at X-S/c 'A'

$$\boxed{W_A \times Y_{AB} = W_B \times Y_{BA}}$$

if $W_A = W_B \Rightarrow \boxed{Y_{AB} = Y_{BA}}$

Eg:-



if $Y_B = \delta = \delta_{BA}$

det $Y_A = Y_{AB} = ?$

$W_A \times Y_{AB} = Y_B \times Y_{BA}$

$3W \times Y_{AB} = 4W \times \delta$

$Y_{AB} = \frac{4}{3}\delta = \underline{\underline{Y_A}}$

Castigliano's Theorem:—

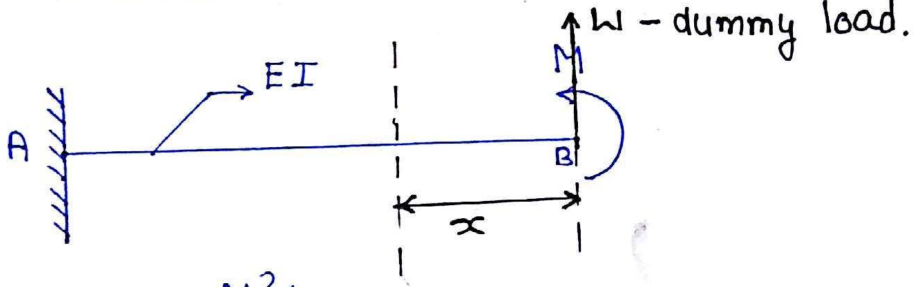
let U = S.E. due to B.M.

$$U = \int_a^b \frac{(M_{x-x})^2\, dx}{2(E\, I_{N.A})_{x-x}}$$

As per this theorem

$$\boxed{\theta_A = \frac{\partial U}{\partial M_A}} \quad : \quad \boxed{Y_A = \frac{\partial U}{\partial W_A}}$$

Eq.



$$U = \frac{M^2 L}{2E I_{N.A.}}$$

$$\theta_B = \frac{\partial U}{\partial M_B} = \frac{\partial}{\partial M}\left[\frac{M^2 L}{2E I_{N.A.}}\right]$$

$$\theta_B = \frac{ML}{E I_{N.A.}}$$

To det. Y_B, introduce a dummy load (Conc. pt.) such that it will cause B.M. in same dirn. at B

$$M_{x-x} = M + Wx$$

$$U_{x-x} = \int_0^L \frac{(M+Wx)^2\, dx}{2EI}$$

$$U = \frac{1}{2EI} \int_0^L \left(M^2 + W^2x^2 + 2MWx\right) dx$$

$$U = \frac{1}{2EI} \left[M^2L + \frac{W^2L^3}{3} + MWL^2 \right]$$

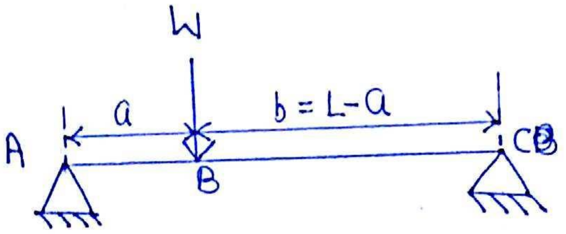
$$Y_B = \frac{\partial U}{\partial W_B} = \frac{1}{2EI} \frac{\partial}{\partial W} \left[M^2L + \frac{W^2L^3}{3} + MWL^2 \right]$$

$$Y_B = \frac{1}{2EI} \left[0 + \frac{2WL^3}{3} + ML^2 \right]$$

To get actual Y_B, Sub $W = 0$

$$Y_B = \frac{ML^2}{2EI}$$

Eg.

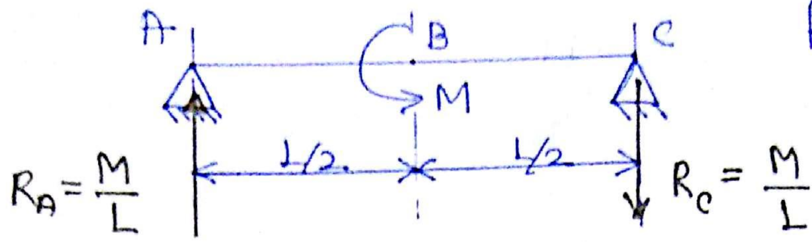


$U = U_{AB} + U_{BC}$

$$U = \frac{W^2a^2b^2}{6EI_{N.A.}L}$$ $\Longrightarrow$ Go back & Check notes "strain energy"

$$Y_B = \frac{\partial U}{\partial W} = \frac{Wa^2b^2}{3EI_{N.A.}L}$$

Eg.



find $\theta_B = ?$

$$U = U_{AB} + U_{BC} \qquad M_{x-x} = \frac{M}{L}x$$

$$U = 2U_{AB} \qquad U_{x-x} = \int_0^x \frac{(M_{x-x})^2 dx}{2EI}$$

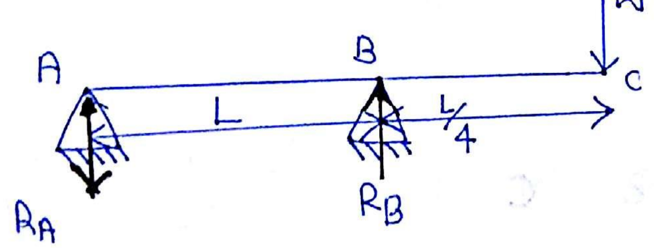
$$U = 2\int_0^{L/2} \frac{\left(\frac{M}{L}x\right)^2 dx}{2EI}$$

$$U = \cancel{2}\ \frac{M^2}{EIL^2}\ \frac{(L/2)^3}{3}$$

$$U = \frac{M^2 L}{24EI}$$

$$\theta_B = \frac{\partial U}{\partial M} = \frac{2ML}{24EI} = \frac{ML}{12EI}$$

Eg



$Y_C = ?$

$$R_A + R_B = W \quad \text{—(1)}$$

$$\Sigma M_A = 0 \Rightarrow W\left(L + \frac{L}{4}\right) - R_B L = 0$$

$$\frac{5WL}{4} = R_B L$$

$$\Rightarrow R_B = \frac{5W}{4}(\uparrow)\ ;\ R_A = \frac{W}{4}(\downarrow)$$

$$U = U_{AB} + U_{BC}$$

$$U = \int_0^L \frac{\left(-\frac{Wx}{4}\right)^2 dx}{2EI} + \int_0^{L/4} \frac{(Wx)^2}{2EI}\, dx$$

$$U = \frac{W^2}{32EI}\,\frac{L^3}{3} + \frac{W^2}{2EI} \times \cancel{\frac{L^3/4}{3}}\,\frac{(L/4)^3}{3}$$

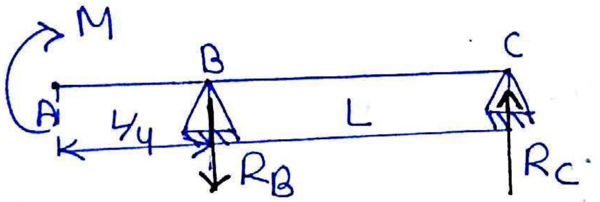
$$\cancel{U = \frac{W^2L^3}{EI}\left(\frac{1}{96} + \frac{1}{6}\right)}$$

$$\cancel{\frac{1+16}{96}}$$

$$U = \frac{5}{384}\,\frac{W^2L^3}{EI}$$

$$Y_C = \frac{dU}{\partial W} = \frac{5\,WL^3}{192\,EI}$$

Eg



$\theta_A = ?$

$$R_C = R_B = \frac{M}{5L/4} \Rightarrow R_C = R_B = \frac{4M}{5L}$$

$$U = U_{AB} + U_{BC}$$

$$U = \frac{M^2(L/4)}{2EI} + \int_0^L \frac{(R_C x)^2 dx}{2EI}$$

$$U = \frac{M^2 L}{8EI} + \frac{R_C^2}{2EI}\,\frac{L^3}{3}$$

$$U = \frac{M^2L}{8EI} + \left(\frac{4M}{5L}\right)^2 \frac{L^3}{6EI}$$

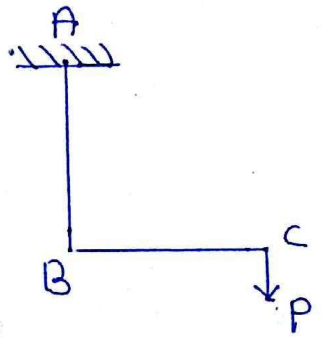
$$U = \frac{M^2L}{EI}\left(\frac{1}{8} + \frac{16}{150}\right)$$

$$U = \frac{M^2L}{EI}\left(\frac{150+128}{8\times150}\right)$$

$$U = \frac{\overset{139}{\cancel{278}}}{\underset{600}{\cancel{1200}}}\,\frac{M^2L}{EI} = \frac{139}{600}\,\frac{M^2L}{EI}$$

$$\theta_A = \frac{139}{300}\,\frac{ML}{EI}$$

Eg.



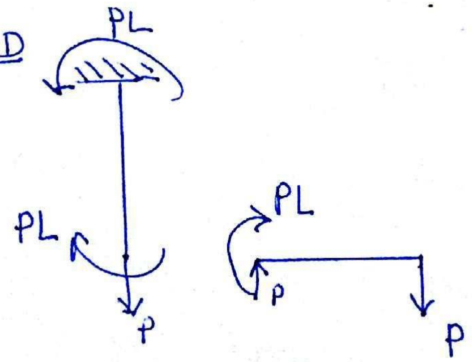
$$U = U_{AB} + U_{BC}$$

$$U = \left(\frac{M^2L}{2EI}\right)_{AB} + \int_0^L \frac{(Px)^2\,dx}{2EI}$$

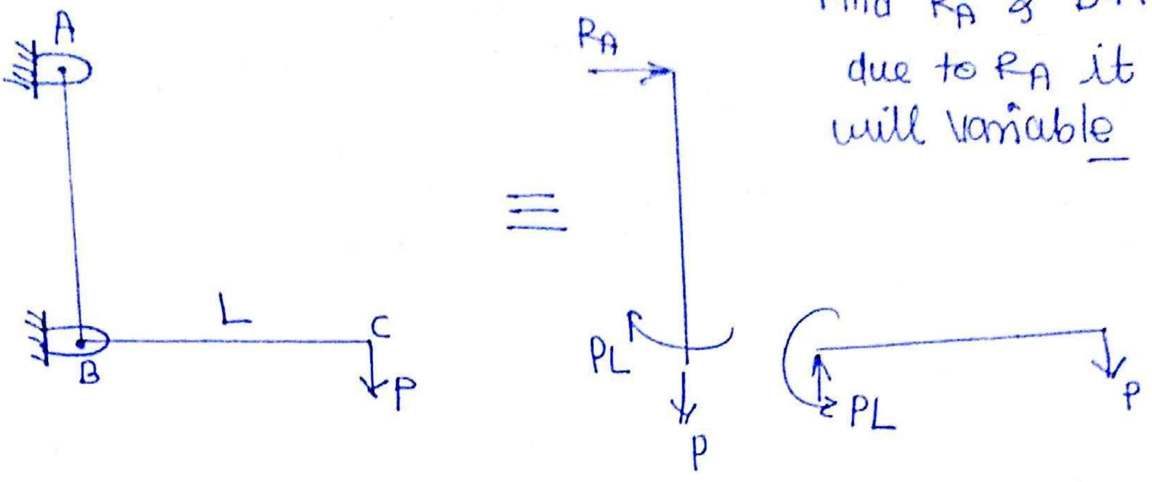
$$U = \frac{(PL)^2L}{2EI} + \frac{P^2L^3}{6EI} = \frac{2}{3}\,\frac{P^2L^3}{EI}$$

$$Y_c = \frac{\partial U}{\partial P} = \frac{4PL^3}{3EI}$$

FBD

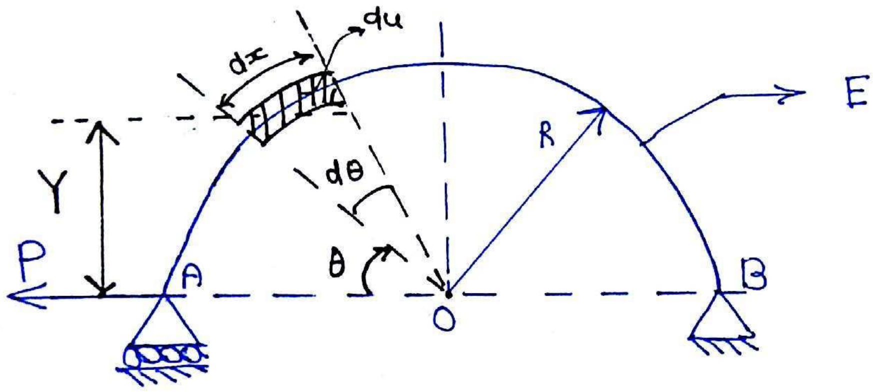


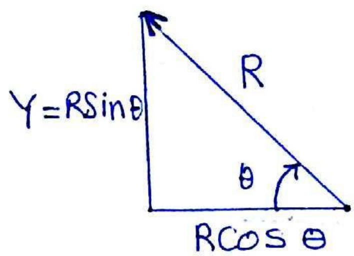
Eg
Gate
see sol^n



Find R_A & B.M. due to R_A it will variable

Quest Determine axial displacment at A, Semicircular ring.





$$\theta = 0 \text{ to } \pi$$

$$M_{x-x} = PY = PR\sin\theta$$

$$dx = R\,d\theta$$

$$U = \int_0^{\pi} \frac{(PR\sin\theta)^2 (R\,d\theta)}{2EI_{N.A.}} = \frac{P^2R^3}{2EI}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$U = \frac{P^2R^3}{2EI}\left(\frac{\pi}{2}\right) = \frac{\pi P^2R^3}{4EI_{N.A.}}$$

$$\delta_A = \frac{\partial u}{\partial p} \Rightarrow \delta_A = \frac{\pi P R^3}{2EI_{N.A.}}$$