

Chapter 9

Sequence and Series

Exercise 9.4

Question 1: Find the sum to n terms of the series $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots$

Answer 1:

The given series is $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots$ n^{th} term, $a_n = n(n + 1)$

$$S_n = \sum_{k=1}^n a_k = \sum_{k=1}^n k(k + 1)$$

$$= \sum_{k=1}^n k^2 + \sum_{k=1}^n k$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2} \left[\frac{2n+1}{3} + 1 \right]$$

$$= \frac{n(n+1)}{2} \left(\frac{2n+4}{3} \right)$$

$$= \frac{n(n+1)(n+2)}{3}$$

Question 2: Find the sum to n terms of the series $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots$

Answer 2:

The given series is $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots$ n^{th} term

$$= a_n = n(n + 1)(n + 2)$$

$$= (n^2 + n)(n + 2)$$

$$= n^3 + 3n^2 + 2n$$

$$\begin{aligned}
S_n &= \sum_{k=1}^n a_k \\
&= \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\
&= \left[\frac{n(n+1)}{2} \right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} \\
&= \left[\frac{n(n+1)}{2} \right]^2 + \frac{n(n+1)(2n+1)}{2} + n(n+1) \\
&= \frac{n(n+1)}{2} \left[\frac{n(n+1)}{2} + 2n + 1 + 2 \right] \\
&= \frac{n(n+1)}{2} \left[\frac{n^2 + n + 4n + 6}{2} \right] \\
&= \frac{n(n+1)}{4} (n^2 + 5n + 6) \\
&= \frac{n(n+1)}{4} (n^2 + 2n + 3n + 6) \\
&= \frac{n(n+1)[n(n+2) + 3(n+2)]}{4} \\
&= \frac{n(n+1)(n-2)(n-3)}{4}
\end{aligned}$$

Question 3: Find the sum to n terms of the series $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \dots$

Answer 3:

The given series is $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \dots$ n^{th} term

$$a_n = (2n + 1) n^2 = 2n^3 + n^2$$

$$\begin{aligned}
S_n &= \sum_{k=1}^n a_k \\
&= \sum_{k=1}^n (2k^3 + k^2) = 2 \sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \\
&= 2 \left[\frac{n(n+1)}{2} \right]^2 + \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

$$\begin{aligned}
&= \frac{n(n+1)}{2} \left[n(n+1) + \frac{2n+1}{3} \right] \\
&= \frac{n(n+1)}{2} \left[\frac{3n^2+3n+2n+1}{3} \right] \\
&= \frac{n(n+1)}{2} \left[\frac{3n^2+5n+1}{3} \right] \\
&= \frac{n(n+1)(3n^2+5n+1)}{6}
\end{aligned}$$

Question 4: Find the sum to n terms of the series $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$

Answer 4:

The given series is $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$

n^{th} term, $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ [By partial fractions]

$$a_1 = \frac{1}{1} - \frac{1}{2}$$

$$a_2 = \frac{1}{2} - \frac{1}{3}$$

$$a_3 = \frac{1}{3} - \frac{1}{4} \dots$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding the above terms column wise, we obtain

$$= a_1 + a_2 + \dots + a_n = \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right] - \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n+1} \right]$$

$$S_n = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

Question 5: Find the sum to n terms of the series $5^2 + 6^2 + 7^2 + \dots + 20^2$

Answer 5:

The given series is $5^2 + 6^2 + 7^2 + \dots + 20^2$ nth term

$$a_n = (n+4)^2 = n^2 + 8n + 16$$

$$S_n = \sum_{k=1}^n a_k = \sum_{k=1}^n (k^2 + 8k + 16)$$

$$= \sum_{k=1}^n k^2 + 8 \sum_{k=1}^n k + \sum_{k=1}^n 16$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{8n(n+1)}{2} + 16n$$

$$16^{\text{th}} \text{ term is } (1 + 4)2 = 202$$

$$S_n = \frac{16(16+1)(2 \times 16+1)}{6} + \frac{8 \times 16 \times (16+1)}{2} + 16 \times 16$$

$$= \frac{(16)(17)(33)}{6} + \frac{(8)(16)(17)}{2} + 16 \times 16$$

$$= 1496 + 1088 + 256$$

$$= 2840$$

$$5^2 + 6^2 + 7^2 + \dots + 20^2 = 2840$$

Question 6: Find the sum to n terms of the series $3 \times 8 + 6 \times 11 + 9 \times 14 + \dots$

Answer 6:

The given series is $3 \times 8 + 6 \times 11 + 9 \times 14 + \dots$ an

$$= (n^{\text{th}} \text{ term of } 3, 6, 9 \dots) \times (n^{\text{th}} \text{ term of } 8, 11, 14 \dots)$$

$$= (3n) (3n + 5)$$

$$= 9n^2 + 15n$$

$$S_n = \sum_{k=1}^n a_k = \sum_{k=1}^n (9k^2 + 15k)$$

$$= 9 \sum_{k=1}^n k^2 + 15 \sum_{k=1}^n k$$

$$= 9 \times \frac{n(n+1)(2n+1)}{6} + 15 \times \frac{n(n+1)}{2}$$

$$= \frac{3n(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2}$$

$$= \frac{3n(n+1)}{2} (2n + 1 + 5)$$

$$= \frac{3n(n+1)}{2} (2n + 6)$$

$$= 3n (n + 1) (n + 3)$$

Question 7: Find the sum to n terms of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$

Answer 7:

The given series is $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$ an

$$= (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(2n^2+3n+1)}{6} = \frac{2^3+3n^2+n}{6}$$

$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_n = \sum_{k=1}^n a_k$$

$$= \sum_{k=1}^n \left[\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k \right]$$

$$= \frac{1}{3} \sum_{k=1}^n k^3 + \frac{1}{2} \sum_{k=1}^n k^2 + \frac{1}{6} \sum_{k=1}^n k$$

$$= \frac{1}{3} \frac{n^2(n+1)^2}{(2)^2} + \frac{1}{2} \times \frac{n(n+1)(2n+1)}{6} + \frac{1}{6} \times \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6} \left[\frac{n(n+1)}{2} + \frac{(2n+1)}{2} + \frac{1}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[\frac{n^2+n+2n+1+1}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[\frac{n^2+n+2n+2}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[\frac{n(n+1)+2(n+1)}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[\frac{(n+1)(n+2)}{2} \right]$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

Question 8: Find the sum to n terms of the series whose n^{th} term is given by $n(n+1)(n+4)$.

Answer 8:

$$a_n = n(n+1)(n+4) = n(n^2 + 5n + 4) = n^3 + 5n^2 + 4n$$

$$S_n = \sum_{k=1}^n a_k = \sum_{k=1}^n k^3 + 5 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k$$

$$= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}$$

$$= \frac{n(n+1)}{2} \left[\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4 \right]$$

$$= \frac{n(n+1)}{2} \left[\frac{3n^2 + 3n + 20n + 10 + 24}{6} \right]$$

$$= \frac{n(n+1)}{2} \left[\frac{3n^2 + 23n + 34}{6} \right]$$

$$= \frac{n(n+1)(3n^2 + 23n + 34)}{12}$$

Question 9: Find the sum to n terms of the series whose nth terms is given by $n^2 + 2^n$

Answer 9:

$$a_n = n^2 + 2^n$$

$$S_n = \sum_{k=1}^n k^2 + 2^k = \sum_{k=1}^n k^2 + \sum_{k=1}^n 2^k \dots (1)$$

$$\text{Consider } \sum_{k=1}^n 2^k = 2^1 + 2^2 + 2^3 + \dots$$

The above series $2, 2^2, 2^3, \dots$ is a G.P. with both the first term and common ratio equal to 2.

$$= \sum_{k=1}^n 2^k = \frac{(2)[(2)^n - 1]}{2 - 1} = 2(2^n - 1) \dots (2)$$

Therefore, from (1) and (2), we obtain

$$S_n = \sum_{k=1}^n k^2 + 2(2^n - 1) = \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

Question 10: Find the sum to n terms of the series whose nth terms is given by

$$(2n - 1)^2$$

Answer 10:

$$a_n = (2n - 1)^2 = 4n^2 - 4n + 1$$

$$S_n = \sum_{k=1}^n a_k = \sum_{k=1}^n (4k^2 - 4k + 1)$$

$$= 4 \sum_{k=1}^n k^2 - 4 \sum_{k=1}^n k + \sum_{k=1}^n 1$$

$$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n$$

$$= n \left[\frac{2(2n^2 + 3n + 1)}{3} - 2(n + 1) + 1 \right]$$

$$= n \left[\frac{4n^2 + 6n + 2 - 6n - 6 + 3}{3} \right]$$

$$= n \left[\frac{4n^2 - 1}{3} \right]$$

$$= \frac{n(2n+1)(2n-1)}{3}$$