EXERCISE 10.3 - VECTOR ALGEBRA

Q No 1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a}\cdot\vec{b} = \sqrt{6}$.

Sol. Let θ be the required angle, then.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\cdot 2} = \frac{\sqrt{2}\sqrt{3}}{\sqrt{3}\sqrt{2}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

Q No. 2 Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

Sol. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$$|\vec{a}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}.$$

and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$$|\vec{b}| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{9+4+1} = \sqrt{14}.$$

If θ is the angle between $\vec{a}$ and $\vec{b}$ then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(1)(3) + (-2)(-2) + (3)(1)}{\sqrt{14}\sqrt{14}} = \frac{3+4+3}{14} = \frac{10}{14} = \frac{5}{7}$$

$$\therefore \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Q No 3 Find projection of $\hat{i} - \hat{j}$ on vector $\hat{i} + \hat{j}$.

Sol. Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$

Then projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$= \frac{(1)(1) + (-1)(1)}{\sqrt{(1)^2 + (1)^2}} = 0$$

Q No. 4 Find projection of $\hat{i} + 3\hat{j} + 7\hat{k}$ on vector $7\hat{i} - \hat{j} + 8\hat{k}$

Sol. Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

Then projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{(1)(7) + (3)(-1) + (7)(8)}{\sqrt{(7)^2 + (-1)^2 + (8)^2}}$

$$= \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

Q.No 5. Show that each of following vector is a unit vector : $\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$, $\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$, $\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$. Also show that they are mutually perpendicular to each other.

Sol. Let $\vec{a} = \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{k} + \frac{6}{7}\hat{k}$

$\vec{b} = \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$

$\vec{c} = \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k}) = \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}$.

Then $|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = \sqrt{\frac{49}{49}} = \sqrt{1} = 1$

$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{-6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = \sqrt{\frac{49}{49}} = \sqrt{1} = 1$

$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{-3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = \sqrt{\frac{49}{49}} = \sqrt{1} = 1$

$\therefore$ $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors.

Also $\vec{a}\cdot\vec{b} = \left(\frac{2}{7}\right)\left(\frac{3}{7}\right) + \left(\frac{3}{7}\right)\left(\frac{-6}{7}\right) + \left(\frac{6}{7}\right)\left(\frac{2}{7}\right)$

$= \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0$

$\vec{b}\cdot\vec{c} = \left(\frac{3}{7}\right)\left(\frac{6}{7}\right) + \left(\frac{-6}{7}\right)\left(\frac{2}{7}\right) + \left(\frac{2}{7}\right)\left(\frac{-3}{7}\right)$

$= \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0$

$\vec{c}\cdot\vec{a} = \left(\frac{6}{7}\right)\left(\frac{2}{7}\right) + \left(\frac{2}{7}\right)\left(\frac{3}{7}\right) + \left(\frac{-3}{7}\right)\left(\frac{6}{7}\right)$

$= \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0$.

$\therefore$ $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$

$\Rightarrow$ $\vec{a} \perp \vec{b}$, $\vec{b} \perp \vec{c}$ and $\vec{c} \perp \vec{a}$.

$\Rightarrow$ $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular to each other.

Q.No. 6 Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$

Sol. Given that $|\vec{a}| = 8|\vec{b}|$ ...... (1)

and $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 8$ ...... (2)

i.e. $\vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b} = 8$

$\Rightarrow \quad |\vec{a}|^2 - |\vec{b}|^2 = 8 \qquad (\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$

$\therefore \left(8|\vec{b}|\right)^2 - \left(|\vec{b}|\right) = 8 \qquad (\text{using (i)})$

$\Rightarrow \quad 64|\vec{b}|^2 - |\vec{b}|^2 = 8$

$\Rightarrow \quad 63|\vec{b}|^2 = 8 \Rightarrow |\vec{b}|^2 = \frac{8}{63}$

$\Rightarrow \quad |\vec{b}| = \sqrt{\frac{8}{63}} = \sqrt{\frac{4}{9}\times\frac{2}{7}} = \frac{2}{3}\sqrt{\frac{2}{7}} \quad (\because |\vec{b}| \neq 0)$

Also $|\vec{a}| = |\vec{b}|\cdot 8 = 8 \times \frac{2}{3}\sqrt{\frac{2}{7}} = \frac{16}{3}\sqrt{\frac{2}{7}}$

QNo7 Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$

Sol. $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$

$= (3\vec{a})\cdot(2\vec{a}) + (3\vec{a})\cdot(7\vec{b}) + (-5\vec{b})\cdot(2\vec{a}) + (-5\vec{b})\cdot(7\vec{b})$

$= 6\vec{a}\cdot\vec{a} + 21\vec{a}\cdot\vec{b} - 10\vec{b}\cdot\vec{a} - 35\vec{b}\cdot\vec{b}$

$= 6|\vec{a}|^2 + 11\vec{a}\cdot\vec{b} - 35|\vec{b}|^2 \qquad [\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$

QNo 8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$ having same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

Sol. Given that $|\vec{a}| = |\vec{b}|$ and $\vec{a}\cdot\vec{b} = \frac{1}{2}$

If θ is angle between $\vec{a}$ and $\vec{b}$ then

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos\theta$

$\therefore \quad \frac{1}{2} = |\vec{a}||\vec{a}|\cos 60^\circ \qquad [\because |\vec{a}| = |\vec{b}|]$

ie $\quad \frac{1}{2} = |\vec{a}|^2\cdot\frac{1}{2} \qquad$ ie $|\vec{a}|^2 = 1$

$\Rightarrow |\vec{a}| = 1 \qquad [\because |\vec{a}| \neq 0]$

$\therefore \quad |\vec{a}| = |\vec{b}| = 1$

QNo 9. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 12$.

Sol. Since $\vec{a}$ is a unit vector

$\therefore \quad |\vec{a}| = 1$

Now $\quad (\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 12$

$\Rightarrow \quad \vec{x}\cdot\vec{x} - \vec{a}\cdot\vec{x} + \vec{x}\cdot\vec{a} - \vec{a}\cdot\vec{a} = 12$

$\Rightarrow \quad |\vec{x}|^2 - |\vec{a}|^2 = 12 \qquad [\because \vec{a}\cdot\vec{x} = \vec{x}\cdot\vec{a}]$

$\Rightarrow \quad |\vec{x}|^2 - (1)^2 = 12$

$\Rightarrow \quad |\vec{x}|^2 = 13$

$\Rightarrow \quad |x| = \sqrt{13}$

Q No 10. If $\vec{a} = 2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, find value of λ.

Sol. Since $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$

$\Rightarrow (\vec{a}+\lambda\vec{b}) \cdot \vec{c} = 0$

ie $\left[(2\hat{i}+2\hat{j}+3\hat{k}) + \lambda(-\hat{i}+2\hat{j}+\hat{k})\right] \cdot [3\hat{i}+\hat{j}] = 0$

ie $\left[(2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}\right] \cdot [3\hat{i}+\hat{j}] = 0$

ie $(2-\lambda)(3) + (2+2\lambda)(1) + (3+\lambda)(0) = 0$

ie $6-3\lambda+2+2\lambda = 0$

ie $8-\lambda = 0 \Rightarrow \lambda = 8.$

Q No. 11. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|.\vec{a}$ for any two non-zero vectors $\vec{a}$ and $\vec{b}$.

Sol. $\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right) \cdot \left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$

$= |\vec{a}|^2(\vec{b}\cdot\vec{b}) + |\vec{b}||\vec{a}|(\vec{a}\cdot\vec{b}) - |\vec{a}||\vec{b}|(\vec{b}\cdot\vec{a}) - |\vec{b}|^2(\vec{a}\cdot\vec{a})$

$= |\vec{a}|^2|\vec{b}|^2 + 0 - |\vec{b}|^2|\vec{a}|^2 \qquad [\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$

$= 0$

$\therefore |\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$ are mutually $\perp r$.

Q No 12. If $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

Sol. $\because \vec{a}\cdot\vec{a} = 0$ the $|\vec{a}|^2 = 0$

$\Rightarrow |\vec{a}| = 0 \Rightarrow \vec{a} = 0$

Hence $\vec{a}\cdot\vec{b} = 0$ what ever $\vec{b}$ may be.

This means that nothing can be concluded about $\vec{b}$ from given data, $\vec{b}$ can be any vector.

Q.No. 13 If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$

Sol. We are given that $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$

and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}\cdot\vec{0}$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$ $\left[\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a},\ \vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{b},\ \vec{c}\cdot\vec{a}=\vec{a}\cdot\vec{c}\right]$

$\Rightarrow (1)^2+(1)^2+(1)^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-3$

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-3/2.$

Q.No. 14. If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$, but the converse need not be true. Justify your answer with an example.

Sol. Surely $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0} \Rightarrow \vec{a}\cdot\vec{b}=0$

However the converse is not true

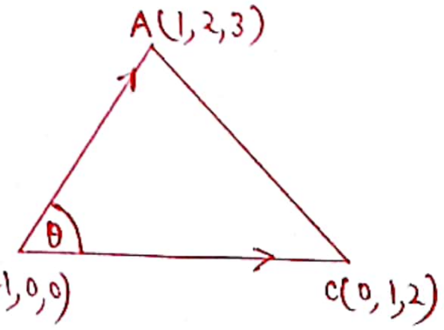
i.e. If $\vec{a}\cdot\vec{b}=0$ then it may not imply that either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$

e.g. Let $\vec{a}=\hat{i}$ $\vec{b}=\hat{j}$

then $\vec{a}\neq\vec{0}$ and $\vec{b}\neq\vec{0}$

but $\vec{a}\cdot\vec{b}=\hat{i}\cdot\hat{j}=0$

Q.No. 15. If the vertices A, B, C of ΔABC are $A(1,2,3)$, $B(-1,0,0)$ and $C(0,1,2)$ resp., then find $\angle ABC$ [$\angle ABC$ is angle between the vectors $\vec{BA}$ and $\vec{BC}$]



Sol. Now $\vec{BA}$ = P.V of A − P.V. of B

$=(\hat{i}+2\hat{j}+3\hat{k})-(-\hat{i}+0\hat{j}+0\hat{k})$

$=2\hat{i}+2\hat{j}+3\hat{k}$

$\vec{BC}$ = P.V of $\vec{C}$ − P.V of $\vec{B}$

$=(0\hat{i}+\hat{j}+2\hat{k})-(-\hat{i}+0\hat{j}+0\hat{k})=1\hat{i}+1\hat{j}+2\hat{k}$

Then $\cos(\angle ABC) = \dfrac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}|\,|\vec{BC}|}$

$$= \frac{(2\hat{i}+2\hat{j}+3\hat{k})\cdot(1\hat{i}+1\hat{j}+2\hat{k})}{\sqrt{(2)^2+(2)^2+(3)^2}\,\sqrt{(1)^2+(1)^2+(2)^2}}$$

$$= \frac{(2)(1)+(2)(1)+(3)(2)}{\sqrt{4+4+9}\,\sqrt{1+1+4}} = \frac{2+2+6}{\sqrt{17}\,\sqrt{6}} = \frac{10}{\sqrt{102}}$$

$$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Q No 16 Show that the points $A(1,2,7)$, $B(2,6,3)$, $C(3,10,-1)$ are collinear.

Sol.

$\vec{AB}$ = P.V of $\vec{B}$ − P.V of $\vec{A}$

$= (2\hat{i}+6\hat{j}+3\hat{k}) - (\hat{i}+2\hat{j}+7\hat{k})$

$= \hat{i}+4\hat{j}-4\hat{k}$

$\vec{AC}$ = P.V of $\vec{C}$ − P.V. of $\vec{A}$

$= (3\hat{i}+10\hat{j}-\hat{k}) - (\hat{i}+2\hat{j}+7\hat{k})$

$= 2\hat{i}+8\hat{j}-8\hat{k}$

$= 2(\hat{i}+4\hat{j}-4\hat{k})$

$= 2\,\vec{AB}$

$\therefore$ $\vec{AC}$ and $\vec{AB}$ are scalar multiples of each other.

$\Rightarrow$ A, B and C are collinear.

Q No 17 Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

Sol. Let $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}-3\hat{j}-5\hat{k}$, $\vec{c} = 3\hat{i}-4\hat{j}-4\hat{k}$

We notice that $\vec{a}+\vec{b} = (2\hat{i}-\hat{j}+\hat{k})+(\hat{i}-3\hat{j}-5\hat{k})$

$= 3\hat{i}-4\hat{j}-4\hat{k}$

$= \vec{c}$

and $\vec{a}\cdot\vec{b} = (2)(1)+(-1)(-3)+(1)(-5)$

$= 2+3-5 = 0.$

$\Rightarrow \vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\therefore$ $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right angled triangle.

Q No 18. If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ a non-zero scalar, then $\lambda\vec{a}$ is unit vector if.

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = 1/|\lambda|$

Sol.

$\lambda\vec{a}$ is a unit vector

if $|\lambda\vec{a}| = 1$.

ie if $|\lambda||\vec{a}| = 1$.

ie if $|\lambda| a = 1$

ie if $a = \frac{1}{|\lambda|}$

$\Rightarrow$ (D) is the correct option.

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