JEE ADVANCE - 2016 (Paper 1)

37. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\,sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\,tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals,

(A) $2(sec\theta - tan\theta)$ (B) $2sec\theta$ (C) $-2tan\theta$ (D) 0

Sol. : $x^2 - 2x\,sec\theta + 1 = 0$

$\therefore\ x^2 - 2x\,sec\theta + sec^2\theta - tan^2\theta = 0$

$\therefore\ (x - sec\theta)^2 = tan^2\theta$

$\therefore\ x - sec\theta = \pm\,tan\theta$

$\therefore\ x = sec\theta \pm tan\theta$. Let $\alpha_1 = sec\theta - tan\theta$, $\beta_1 = sec\theta + tan\theta$

$\alpha_1 > \beta_1 \iff sec\theta - tan\theta > sec\theta + tan\theta$

$\iff tan\theta < 0$, which is true, since $\frac{-\pi}{6} < \theta < \frac{-\pi}{12}$.

$\therefore\ \alpha_1 = sec\theta - tan\theta;\ \beta_1 = sec\theta + tan\theta$

$\therefore\ x^2 + 2x\,tan\theta - 1 = 0$

$\therefore\ (x + tan\theta)^2 = sec^2\theta$

$\therefore\ x + tan\theta = \pm\,sec\theta$. Let $\alpha_2 = sec\theta - tan\theta$, $\beta_2 = -sec\theta - tan\theta$

Also, $\alpha_2 > \beta_2 \iff -tan\theta + sec\theta > -tan\theta - sec\theta$

$\iff 2sec\theta > 0$, which is true. Since $\frac{-\pi}{6} < \theta < \frac{-\pi}{12}$.

$\therefore\ \alpha_2 = -tan\theta + sec\theta$

$\beta_2 = -tan\theta - sec\theta$

Hence, $\alpha_1 + \beta_2 = sec\theta - tan\theta - tan\theta - sec\theta$

$= -\,2tan\theta$ **Ans. (C)**

38. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

(A) 380 (B) 320 (C) 260 (D) 95

Sol. : If one day boy is selected, then the number of ways of selection = ${}^4C_1 \cdot {}^6C_3$

If no boy is selected, then the number of ways of selection = ${}^4C_0 \cdot {}^6C_4$

A captain can be selected in ${}^4C_1$ ways.

$\therefore$ Required number of ways $= {}^4C_1 \cdot {}^6C_3 \cdot {}^4C_1 + {}^4C_0 \cdot {}^6C_4 \cdot {}^4C_1$

$= 4 \cdot 20 \cdot 4 + 1 \cdot 15 \cdot 4$

$= 320 + 60 = 380$ **Ans. (A)**

39. Let $S = \left\{x \mid x \in (-\pi, \pi); x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\, sec\, x + cosec\, x + 2(tan\, x - cot\, x) = 0$ in the set S is equal to,

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

Sol. : $\sqrt{3}\, sec\, x + cosec\, x + 2(tan\, x - cot\, x) = 0$

$\therefore \quad (\sqrt{3}\, sin\, x + cos\, x) + 2(sin^2 x - cos^2 x) = 0$

$\therefore \quad \frac{\sqrt{3}}{2} sin\, x + \frac{1}{2} cos\, x = cos^2 x - sin^2 x$

$\therefore \quad cos\left(x - \frac{\pi}{3}\right) = cos\, 2x$

$\therefore \quad 2x = 2n\pi \pm \left(x - \frac{\pi}{3}\right); \quad n \in Z$

$\therefore \quad x = \frac{2n\pi}{3} + \frac{\pi}{9}; \quad n \in Z$ or $x = 2n\pi - \frac{\pi}{3}; \quad n \in Z$

$\therefore \quad x = -\frac{\pi}{3}, -\frac{5\pi}{9}, \frac{\pi}{9}, \frac{7\pi}{9}$ $\qquad x \in (-\pi, \pi)$

$\therefore \quad \Sigma x_i = 0$ **Ans. (C)**

40. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20 % and plant T_2 produces 80 % of the total computers produced. 7 % of computers produced in the factory turn out to be defective. It is known that

P(computer turns out to be defective given that it is produced in plant T_1)

= 10P(computer turns out to be defective given that it is produced in plant T_2).

where P(E) denotes the probability of an event E. A computer produced in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$ (C) $\frac{78}{93}$ (D) $\frac{75}{83}$

Sol. : E_1 : The computer is produced in plant T_1

E_2 : The computer is produced in plant T_2

A : The computer is defective.

$P(E_1) = \frac{1}{5}; P(E_2) = \frac{4}{5}; P(A) = \frac{7}{100}$.

Let, $P(A \mid E_2) = x$. Then $P(A \mid E_1) = 10x$

$\therefore \quad P(A) = P(A \mid E_1) \cdot P(E_1) + P(A \mid E_2) \cdot P(E_2)$

$\therefore \quad \frac{7}{100} = \frac{1}{5} \cdot (10x) + \frac{4}{5} \cdot x$

$\therefore \quad 7 = 280x$

$\therefore \quad x = \frac{1}{40}$

$\therefore \quad P(A \mid E_2) = \frac{1}{40}$

$$\text{Now, } P(E_2 \mid \bar{A}) = \frac{P(E_2 \cap \bar{A})}{P(\bar{A})}$$

$$= \frac{P(E_2) - P(E_2 \cap A)}{1 - P(A)}$$

$$= \frac{P(E_2) - P(E_2) \cdot P(A \mid E_2)}{1 - P(A)}$$

$$= \frac{P(E_2) \cdot (1 - P(A \mid E_2))}{1 - P(A)}$$

$$= \frac{\frac{4}{5}\left(1 - \frac{1}{40}\right)}{1 - \frac{7}{100}}$$

$$= \frac{\frac{4}{5} \cdot \frac{39}{40}}{\frac{93}{100}}$$

$$= \frac{78}{93}$$

Ans. (C)

Another method :

E_1 : Computer is produced by plant T_1 ; E_2 : Computer is produced by plant T_2

A : Computer is defective.

Now, $P(A \mid E_1) = 10P(A \mid E_2)$

$$\therefore \frac{P(A \cap E_1)}{P(A \cap E_2)} = \frac{P(E_1)}{P(E_2)} 10 = \frac{10 \cdot \frac{1}{5}}{\frac{4}{5}} = \frac{5}{2}$$

Let $P(E_2 \cap \bar{A}) = \frac{x}{100}$

$\therefore P(E_2 \cap A) = \frac{80 - x}{100}$

$P(E_1 \cap A) = \frac{x - 73}{100}$ $(P(A) - P(E_2 \cap A))$

$$\therefore \frac{x - 73}{100} = \frac{5}{2}\left(\frac{80 - x}{100}\right)$$

$\therefore 2x - 146 = 400 - 5x$

$\therefore 7x = 546$

$\therefore x = 78$

$$\therefore P(E_2 \mid \bar{A}) = \frac{P(E_2 \cap \bar{A})}{P(\bar{A})} = \frac{78}{93}$$

Ans. (C)

41. The least value of $\alpha \in R$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

Sol. : $4\alpha x^2 + \frac{1}{x} \geq 1, x > 0$

Let $f(x) = 4\alpha x^2 + \frac{1}{x}$. So, $f'(x) = 8\alpha x - \frac{1}{x^2}$

$f'(x) = 0$ at $x = \frac{1}{2\alpha^{\frac{1}{3}}}$

Now, $f(x) = \frac{4\alpha x^3 + 1}{x}$

$\therefore \quad f\left(\frac{1}{2\alpha^{\frac{1}{3}}}\right) = \left(\frac{4\alpha}{8\alpha} + 1\right) 2\alpha^{\frac{1}{3}} = 3\alpha^{\frac{1}{3}} \geq 1$

$\therefore \quad (27\alpha) \geq 1$

$\therefore \quad \alpha \geq \frac{1}{27}$

$\therefore$ Least value of $\alpha = \frac{1}{27}$.

Now, $f''(x) = 8\alpha + \frac{2}{x^3} > 0$

$\therefore$ $f(x)$ is minimum for $x = \frac{1}{2\alpha^{\frac{1}{3}}}$ and greater than 1. **Ans. (C)**

SECTION 2

42. Consider a pyramid OPQRS located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin and $\overrightarrow{\mathbf{OP}}$ and $\overrightarrow{\mathbf{OR}}$ along the X-axis and the Y-axis respectively. The base OPQR of the pyramid is a square with OP = 3. The point S is directly above the mid-point T of diagonal $\overrightarrow{\mathbf{OQ}}$ such that TS = 3. Then,

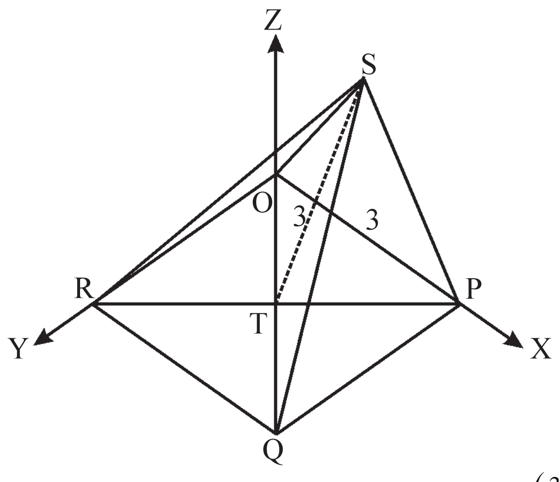
(A) the acute angle between $\overrightarrow{\mathbf{OQ}}$ and $\overrightarrow{\mathbf{OS}}$ is $\frac{\pi}{3}$

(B) the equation of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing $\overrightarrow{\mathbf{RS}}$ is $\sqrt{\frac{15}{2}}$

Sol. :



Points O, P, Q, R, S are (0, 0, 0), (3, 0, 0), (3, 3, 0), (0, 3, 0) and $\left(\frac{3}{2}, \frac{3}{2}, 3\right)$ respectively.

$S = \left(\frac{3}{2}, \frac{3}{2}, 3\right)$.

(A) $\overrightarrow{\mathbf{OQ}} = 3\hat{i} + 3\hat{j}$, $\overrightarrow{\mathbf{OS}} = \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} + 3\hat{k}$

$$\therefore \quad cos\theta = \frac{(3,3,0)\cdot\left(\frac{3}{2},\frac{3}{2},3\right)}{\sqrt{9+9+0}\cdot\sqrt{\frac{9}{4}+\frac{9}{4}+9}} = \frac{9}{3\sqrt{2}\cdot 3\sqrt{\frac{3}{2}}} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad \theta = cos^{-1}\frac{1}{\sqrt{3}}$$

(B) $\overrightarrow{\mathbf{OQ}} \times \overrightarrow{\mathbf{OS}} = (1, 1, 0) \times (1, 1, 2) = (2, -2, 0)$

$\therefore$ The equation of the plane through O, Q, S is $2(x - 0) - 2(y - 0) = 0$.

$\therefore \quad x - y = 0$

(C) The length of the perpendicular to $x - y = 0$ from (3, 0, 0) is $\frac{3-0+0}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

(D) $\overleftrightarrow{\text{RS}} : \frac{x-0}{\frac{3}{2}} = \frac{y-3}{-\frac{3}{2}} = \frac{z-0}{3}$

$$\therefore \quad \frac{x}{1} = \frac{y-3}{-1} = \frac{z-0}{2} = k \text{ (say).}$$

$\therefore$ The perpendicular distance of O from $\overleftrightarrow{\text{RS}} = |\overrightarrow{\mathbf{OR}} \times \hat{l}\,|$

where, O = (0, 0, 0), R = (0, 3, 0), l = (1, –1, 2)

$\therefore \quad \overrightarrow{\mathbf{OR}} = (0, 3, 0)$, $\overrightarrow{\mathbf{OR}} \times \hat{l} = (6, 0, -3)$

$\therefore$ Distance $= \sqrt{\frac{45}{6}} = \sqrt{\frac{15}{2}}$ **Ans. (B), (C), (D)**

43. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then,

(A) $\lim_{x \to 0+} f'\left(\frac{1}{x}\right) = 1$

(B) $\lim_{x \to 0+} xf\left(\frac{1}{x}\right) = 2$

(C) $\lim_{x \to 0+} x^2 f'(x) = 0$

(D) $|f(x)| \leq 2$ for all $x \in (0, 2)$

Sol. : $f'(x) = 2 - \frac{f(x)}{x}$

$$\therefore \quad f'(x) + \frac{f(x)}{x} = 2. \text{ So, } \frac{dy}{dx} + \frac{y}{x} = 2$$

$\therefore \quad x\,dy + y\,dx = 2x\,dx$

$\therefore \quad xy = x^2 + c$, $c \neq 0$ as $f(1) \neq 1$ $\quad (1 \cdot y(1) = 1 + c \neq 1 \Rightarrow c \neq 0)$

$$\therefore \quad y = x + \frac{c}{x} = f(x)$$

(A) $\lim_{x \to 0+} f'\left(\frac{1}{x}\right) = \lim_{x \to 0+} (1 - cx^2) = 1$. (A) is true.

(B) $\lim_{x \to 0+} xf\left(\frac{1}{x}\right) = \lim_{x \to 0+} (1 + cx^2) = 1$. (B) is not true.

(C) $\lim_{x \to 0+} x^2 f'(x) = \lim_{x \to 0+} (x^2 - c) = -c \neq 0$. (C) is not true.

(D) for $c \neq 0$, $f(x)$ is unbounded function $x \in (0, 2)$. (D) is not true. **Ans. (A)**

44. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in R$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then

(A) $\alpha = 0, k = 8$ (B) $4\alpha - k + 8 = 0$ (C) $\det(P\,\text{adj}(Q)) = 2^9$ (D) $\det(Q\,\text{adj}(P)) = 2^{13}$

Sol. : $P\left(\frac{Q}{k}\right) = 1$

$P^{-1} = \left(\frac{Q}{k}\right)$

$(P^{-1})_{23} = \frac{q_{23}}{k} = -\frac{1}{8}$

$-\frac{(3\alpha + 4)}{20 + 12\alpha} = -\frac{1}{8}$. So, $\alpha = -1$

$\therefore$ $\det(P) = 20 + 12\alpha = 8$

$\det(P)\left(\det\left(\frac{Q}{k}\right)\right) = 1$

$\frac{8\det(Q)}{k^3} = 1 \Rightarrow \det Q = \frac{k^3}{8}$

$\therefore$ $\frac{k^3}{8} = \frac{k^2}{2}$. So, $k = 4$. Hence $\det(Q) = 8$

$4\alpha - k + 8 = -4 - 4 + 8 = 0$

$\therefore$ B is true.

$\det(P\,\text{adj}(Q)) = \det P \cdot \det \text{adj} Q$

$= \det P (\det Q)^2 = 8 \times 8^2 = 2^9$

$\det(Q \cdot \text{adj } P) = \det Q (\det P)^2 = 8 \times 8^2 = 2^9$ **Ans. (B), (C)**

Second Method :

$PQ = kI$

$\therefore |P| \cdot |Q| = k^3 \Rightarrow |P| = 2k$ as $|Q| = \frac{k^2}{2}$ ($|P| = \det P$)

$\therefore\ 12\alpha + 20 = 2k$. So, $k = 6\alpha + 10 \Rightarrow 6\alpha - k = -10$ ...(i)

Now, $P\left(\frac{Q}{k}\right) = I$

$\therefore\ \frac{Q}{k} = P^{-1} = \frac{1}{|P|} \cdot \text{adj}P$

$\therefore\ |P|Q = k\,\text{adj}P$

$\therefore\ 2kQ = k\,\text{adj}P$

$\therefore\ Q = \frac{1}{2}\,\text{adj}P$

Now, $q_{23} = -\frac{k}{8} = \frac{-3\alpha - 4}{2}$

$\therefore\ 12\alpha - k = -16$ ...(ii)

By solving (i) and (ii), we get $\alpha = -1$ and $k = 4$

$\therefore$ (A) is not true.

Now, $4\alpha - k + 8 = -4 - 4 + 8 = 0$

$\therefore$ (B) is true.

Now, $|P| = 2k = 8 = 2^3$

$\therefore\ |Q| = \frac{k^2}{2} = 2^3$

$\therefore\ |P\,\text{adj}\,Q| = |P|\,|\text{adj}Q|$

$= |P|\,|Q|^2$

$= 2^3 \cdot (2^3)^2 = 2^9$ is true.

$\therefore$ (C) is correct.

$|Q|\,|\text{adj}\,P| = |Q|\,|P^2| = 2^3 \cdot (2^3)^2 = 2^9$

$\therefore$ (D) is incorrect. **Ans. (B), (C)**

45. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z respectively and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and the area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then

(A) the area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\,\sin\frac{Y}{2}\,\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

Sol. : Here, $x + y + z = 2s$ and $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$

$\therefore \quad \frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{s}{9} = k$ (Say)

$\therefore \quad \Delta = \sqrt{s(s-x)(s-y)(s-z)}$

$= \sqrt{s \cdot \frac{4s}{9} \cdot \frac{3s}{9} \cdot \frac{2s}{9}} = \frac{2\sqrt{6}s^2}{27}$

Also, $r = \frac{\Delta}{s} = \frac{2\sqrt{6}s^2}{27s} = \frac{2\sqrt{6}}{27}s.$ (i)

Now, $s = 9k$, $s - x = 4k$, $s - y = 3k$, $s - z = 2k$

$\therefore \quad x = 5k, y = 6k, z = 7k$

The area of incircle $= \pi r^2 = \frac{8\pi}{3}$. So, $r^2 = \frac{8}{3}$

$\therefore \quad \frac{24}{729} \cdot s^2 = \frac{8}{3}$ from (i)

$\therefore \quad \frac{24}{729} \cdot 81k^2 = \frac{8}{3}$

$\therefore \quad k^2 = 1$

$\therefore \quad k = 1$

$\therefore \quad s = 9, x = 5, y = 6, z = 7$

$\therefore \quad \Delta = \frac{2\sqrt{6}}{27} \cdot 81 = 6\sqrt{6}$

$\therefore$ (A) is true.

Now, $R = \frac{abc}{4\Delta} = \frac{xyz}{4\Delta} = \frac{210}{4(6\sqrt{6})} = \frac{35}{4\sqrt{6}} \neq \frac{35\sqrt{6}}{6}$

$\therefore$ (B) is not true.

Now, $sin\frac{X}{2}\ sin\frac{Y}{2}\ sin\frac{Z}{2} = \frac{r}{4R} = \sqrt{\frac{8}{3}} \cdot \frac{\sqrt{6}}{35} = \frac{4}{35}$

$\therefore$ (C) is true.

$sin^2\left(\frac{X+Y}{2}\right) = sin^2\left(\frac{\pi - Z}{2}\right) = cos^2\ \frac{Z}{2} = \frac{s(s-z)}{xy}$

$= \frac{9 \cdot 2}{30} = \frac{3}{5}$

$\therefore$ (D) is true. **Ans. (A), (C), (D)**

46. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$, $x > 0$, passes through the point (1, 3). Then the solution curve

(A) intersects $y = x + 2$ at exactly one point

(B) intersects $y = x + 2$ at exactly two points

(C) intersects $y = (x + 2)^2$

(D) intersects $y = (x + 3)^2$

Sol. : $[(x + 2)^2 + (x + 2)y]\dfrac{dy}{dx} - y^2 = 0$

Let $y = (x + 2)t$

$\therefore \quad \dfrac{dy}{dx} = (x + 2)\dfrac{dt}{dx} + t$

$\therefore$ The equation is $(x + 2)\,[(x + 2) + y]\dfrac{dy}{dx} = y^2$

$\therefore \quad (x + 2)\,[(x + 2) + (x + 2)t]\left((x + 2)\dfrac{dt}{dx} + t\right) - (x + 2)^2\, t^2 = 0$

$\therefore \quad (x + 2)^2\,(t + 1)\left[(x + 2)\dfrac{dt}{dx} + t\right] - (x + 2)^2\, t^2 = 0$

$\therefore \quad (x + 2)^2\left[(t + 1)(x + 2)\dfrac{dt}{dx} + t^2 + t - t^2\right] = 0$

$\therefore \quad (x + 2)(t + 1)\dfrac{dt}{dx} + t = 0$

$\therefore \quad \dfrac{(t+1)}{t}\,dt + \dfrac{dx}{x+2} = 0$

$\therefore \quad \left(1 + \dfrac{1}{t}\right)dt + \dfrac{dx}{x+2} = 0$

$\therefore \quad t + \log|t| + \log|x + 2| = c$

$\therefore \quad \log|t(x + 2)| + t = c$

$\therefore \quad \log y + \dfrac{y}{x+2} = c$

If $x = 1$, $y = 3$. So, $\log 3 + \frac{3}{3} = c$

$\therefore \quad \log y + \dfrac{y}{x+2} = \log 3e$

$\quad \log y + \dfrac{y}{x+2} = \log 3 + 1$

$\therefore \quad \log \dfrac{y}{3} + \dfrac{y}{x+2} = 1$

If $y = (x + 2)^2$, then $\log \dfrac{(x+2)^2}{3} + x + 2 > 1$ because $\dfrac{(x+2)^2}{3} > \dfrac{4}{3} > 1$

So, $\log \dfrac{(x+2)^2}{3} + x + 2 > \log 1 + x + 2 > 2$

$\therefore$ (C) is not true.

Similarly, for $y = (x + 3)^2$

$\therefore$ (D) is not true. **Ans. (A)**

47. Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then,

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$ (C) $h(0) = 16$ (D) $h(g(3)) = 36$

Sol. : $f(0) = 2, f'(0) = (3x^2 + 3)_{x=0} = 3,$

$g(f(x)) = x$

$\therefore \quad g'(f(x)) \cdot f'(x) = 1$

$\therefore \quad g'(f(0)) \cdot f'(0) = 1$

$\therefore \quad g'(2) \cdot (3) = 1$

$\therefore \quad g'(2) = \frac{1}{3}$

So, (A) is not true.

Now, $h(g(g((x)))) = x$

So, $h(g(g(f(x)))) = f(x)$

$\therefore \quad h(g(x)) = f(x)$

$\therefore \quad h(g(f(x))) = (fof)(x)$

$\therefore \quad h(x) = (fof)(x) = f(f(x))$

$\therefore \quad h(0) = f(f(0)) = f(2)$

$\therefore \quad h(0) = 16$

$\therefore$ (C) is true.

Now, $h(x) = f(f(x))$

$\therefore \quad h'(x) = f'(f(x)) \cdot f'(x)$

$\therefore \quad h'(1) = f'(f(1)) \cdot f'(1)$

$\qquad\qquad = f'(6) \cdot 6$

$\therefore \quad h'(1) = (111)6 = 666$

$\therefore$ (B) is true.

Also, $h(g(3)) = f(3) = 38 \neq 36$

$\therefore$ (D) is not true. **Ans. (B), (C)**

48. The circle $C_1 : x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touch other two circles C_2 and C_3 at R_2 and R_3 respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3 respectively. If Q_2 and Q_3 lie on the Y-axis then,

(A) $Q_2Q_3 = 12$

(B) $R_2R_3 = 4\sqrt{6}$

(C) The area of the triangle OR_2R_3 is $6\sqrt{2}$

(D) The area of the triangle PQ_2Q_3 is $4\sqrt{2}$

Sol. : By solving equation $x^2 + y^2 = 3$ and $x^2 = 2y$, we get the points of intersection.

i.e. $y^2 + 2y - 3 = 0$

$\therefore \quad y = -3$ or 1

$\therefore$ The point of intersection in the first quadrant is $P(\sqrt{2}, 1)$

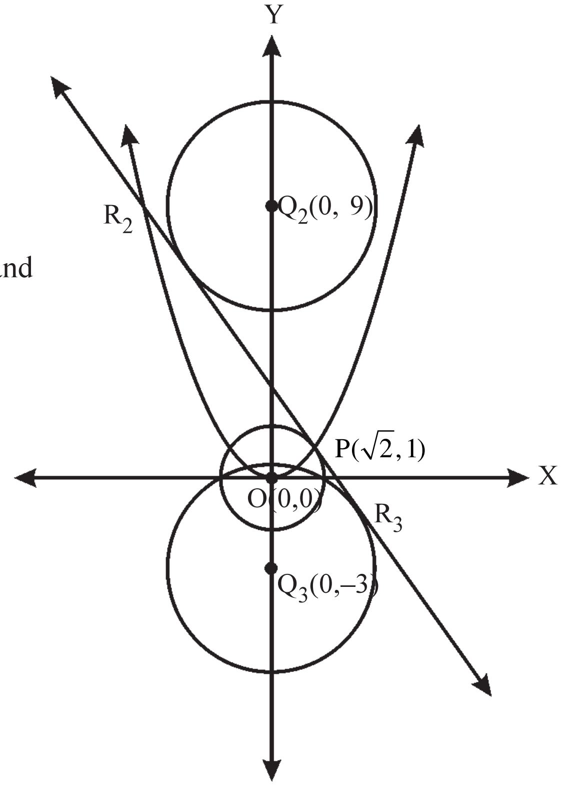
$\therefore$ In the first quadrant, the equation of the tangent at P to the circle is $\sqrt{2}x + y = 3$.

It touches the circles C_2 and C_3. So, the distance of centres $(0, \alpha)$ (lying on Y-axis) is equal to radius $2\sqrt{3}$.

$\therefore \quad \frac{|0+\alpha-3|}{\sqrt{3}} = 2\sqrt{3}$

$\therefore \quad \alpha - 3 = \pm 6$

$\therefore \quad \alpha = 9, -3.$ $Q_2 = (0, 9)$, $Q_3 = (0, -3)$



(A) $Q_2Q_3 = 12$

$\therefore$ (A) is true.

(B) C_2 and C_3 have equations $x^2 + (y-9)^2 = 12$ and $x^2 + (y+3)^2 = 12$.

So, the lengths of tangents from $P(\sqrt{2}, 1)$ are

$\sqrt{2+64-12}$, $\sqrt{2+16-12}$

$= \quad \sqrt{54}, \sqrt{6}$

$= \quad 3\sqrt{6}, \sqrt{6}$

$\therefore \quad R_2R_3 = 3\sqrt{6} + \sqrt{6} = 4\sqrt{6}$

$\therefore$ (B) is true.

(C) The perpendicular distance of origin from

$\sqrt{2}x + y = 3$ is $\frac{|0+0-3|}{\sqrt{3}} = \sqrt{3}$.

$\therefore$ The area of $\Delta OR_2R_3 = \frac{1}{2} \times R_2R_3 \times p$

$= \frac{1}{2} \times 4\sqrt{6} \times \sqrt{3}$

$= 2\sqrt{18} = 6\sqrt{2}$

$\therefore$ (C) is true.

(D) The area of $\Delta PQ_2Q_3 = \frac{1}{2} \times Q_2Q_3 \times \sqrt{2} = \frac{1}{2} \times 12\sqrt{2} = 6\sqrt{2} \neq 4\sqrt{2}$

$\therefore$ (D) is not true. **Ans. (A), (B), (C)**

49. Let $\overleftrightarrow{RS}$ be a diameter of the circle $x^2 + y^2 = 1$, where S is the point (1, 0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at Q. The normal to the circle at P intersects a line drawn through Q parallel to $\overleftrightarrow{RS}$ at point E. Then the locus of E passes through the point(s),

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (C) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Sol. : P = ($cos\theta$, $sin\theta$)

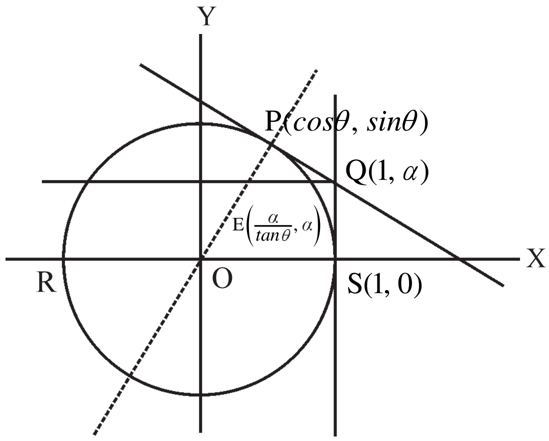
The tangent at P is $x\,cos\theta + y\,sin\theta = 1$

It intersects the tangent at S, i.e. $x = 1$ in $\left(1, \frac{1-cos\theta}{sin\theta}\right)$

The normal at P is passing through the point O. (It is radius)

Its equation is $y = x\,tan\theta$

It intersects the horizontal line $y = \frac{1-cos\theta}{sin\theta}$ in



$$E\left(\frac{1-cos\theta}{sin\theta\, tan\theta}, \frac{1-cos\theta}{sin\theta}\right) = \left(\frac{tan\frac{\theta}{2}}{tan\theta}, tan\frac{\theta}{2}\right)$$

Let $h = \frac{tan\frac{\theta}{2}}{tan\theta}$, $k = tan\frac{\theta}{2}$

$$\therefore\ h = \frac{tan\frac{\theta}{2}(1-tan^2\frac{\theta}{2})}{2tan\frac{\theta}{2}}$$

$$= \frac{1-tan^2\frac{\theta}{2}}{2}$$

$$= \frac{1-k^2}{2}$$

$\therefore$ The locus is $1 - y^2 = 2x$

Given points are $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{4}, \frac{1}{2}\right)$, $\left(\frac{1}{3}, \frac{-1}{\sqrt{3}}\right)$, $\left(\frac{1}{4}, \frac{-1}{2}\right)$.

Obviously, $\left(\frac{1}{3}, \pm\frac{-1}{\sqrt{3}}\right)$ satisfy the equation $1 - y^2 = 2x$.

$\therefore$ (A) and (C) are true. **Ans. (A), (C)**

SECTION 3

50. The total number of distinct $x \in$ R for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

Sol. : $x^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix} + x^6 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 10$

$$\therefore\ x^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 3 & 6 & -2 \end{vmatrix} + x^6 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & 6 \\ 3 & 6 & 24 \end{vmatrix} = 10$$

$\therefore$ $x^3(-4 + 6) + x^6(48 - 36) = 10$

$\therefore$ $2x^3 + 12x^6 = 10$

$\therefore \quad 6x^6 + x^3 - 5 = 0$

$\therefore \quad (6x^3 - 5)(x^3 + 1) = 0$

$\therefore \quad x^3 = \frac{5}{6}$ or $x^3 = -1 \quad \Rightarrow \quad x = \left(\frac{5}{6}\right)^{\frac{1}{3}}$ or $x = -1$

Hence, here are **two** real solutions.

51. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is ....

Sol. : Here $(3n + 1)\ {}^{51}C_3 = \left(\text{Coefficient of } x^2 \text{ in } (1+x)^2 \frac{[(1+x)^{48} - 1]}{1+x-1}\right) + {}^{50}C_2\, m^2$

$\therefore \quad (3n + 1)\ {}^{51}C_3 = (\text{Coefficient of } x^3 \text{ in } (1 + x)^{50} - (1 + x)^2) + {}^{50}C_2\, m^2$

$$\therefore \quad 3n\binom{51}{3} + \binom{51}{3} = \binom{50}{3} + \binom{50}{2}(m^2 - 1) + \binom{50}{2}$$

$$\therefore \quad 3n\,\frac{51}{3}\binom{50}{2} + \binom{51}{3} = \binom{50}{2} + \binom{50}{3} + (m^2 - 1)\binom{50}{2}$$

$$\therefore \quad 51n\binom{50}{2} + \binom{51}{3} = \binom{51}{3} + (m^2 - 1)\binom{50}{2}$$

$$\therefore \quad 51n\binom{50}{2} = (m^2 - 1)\binom{50}{2}$$

$$\therefore \quad n = \frac{m^2 - 1}{51}$$

$\therefore$ The smallest n for which m is an integer is $n = 5$.

52. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}\, dt = 2x - 1$ is

Sol. : $\int_0^x \frac{t^2}{1+t^4}\, dt = 2x - 1$

Let $f(x) = \int_0^x \frac{t^2}{1+t^4}\, dt - 2x + 1$

$\therefore f'(x) = \frac{x^2}{1+x^4} - 2 < 0, \quad \forall x \in [0, 1]$

Now, $f(0) = 1$ and $f(1) = \int_0^1 \frac{t^2}{1+t^4}\, dt - 1$

As $0 \le \frac{t^2}{1+t^4} \le \frac{1}{2}, \quad \forall t \in [0, 1]$

$\therefore \int_0^1 \frac{t^2}{1+t^4}\, dt \le \frac{1}{2} \Rightarrow f(1) < 0$. So, $f(0) > 0$ and $f(1) < 0$.

Also, $f'(x) < 0. \quad \forall x \in [0, 1]$

$\therefore f(x) = 0$ has exactly **one** root in [0, 1].

53. Let $\alpha, \beta \in R$ be such that $\lim_{x \to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha + \beta)$ equals ...

Sol. : $\lim_{x \to 0} \frac{x^2 \sin \beta x}{\alpha x - \sin x} \times \frac{\beta x}{\beta x} = 1$

$$\lim_{x \to 0} \frac{x^3 \beta}{\alpha x - \left[x - \frac{x^3}{\underline{3}} + \frac{x^3}{\underline{5}} - ... \right]} = 1$$

$$\lim_{x \to 0} \frac{x^3 \beta}{(\alpha - 1)x + \frac{x^3}{\underline{3}} - \frac{x^3}{\underline{5}} + ...} = 1$$

$\alpha - 1 = 0, \beta \times 3! = 1$

$\alpha = 1$ and $\beta = \frac{1}{6}$

$6(\alpha + \beta) = 6\left(1 + \frac{1}{6}\right) = 7$ **Ans. (7)**

54. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

Sol. : $z = \omega$

$$P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$P^2 = \begin{bmatrix} (-\omega)^r & (\omega)^{2s} \\ (\omega)^{2s} & \omega^r \end{bmatrix} \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix} = \begin{bmatrix} (-\omega)^{2r} + \omega^{4s} & (-\omega)^r(\omega)^{2s} + \omega^{2s}\omega^r \\ (-\omega)^r \omega^{2s} + \omega^r \omega^{2s} & \omega^{4s} + \omega^{2r} \end{bmatrix}$$

$P^2 = -I$

$\omega^{2r} + \omega^{4s} = -1$ and $\omega^{2s}(-\omega)^r + \omega^{2s}\omega^r = 0$

$\omega^{2r} + \omega^s = -1$

$I + \omega^s + \omega^{2r} = 0$

$r = s = I$

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