CHAPTER XIV.

Binomial Theorem. Any Index.

177. In the last chapter we investigated the Binomial Theorem when the index was any positive integer; we shall now consider whether the formulæ there obtained hold in the case of negative and fractional values of the index.

Since, by Art. 167, every binomial may be reduced to one common type, it will be sufficient to confine our attention to binomials of the form $(1+x)^n$.

By actual evolution, we have

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots\ldots;$$

and by actual division,

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots;$$

[Compare Ex. 1, Art. 60.]

and in each of these series the number of terms is unlimited.

In these cases we have by independent processes obtained an expansion for each of the expressions $(1+x)^{\frac{1}{2}}$ and $(1+x)^{-2}$. We shall presently prove that they are only particular cases of the general formula for the expansion of $(1+x)^n$, where n is any rational quantity.

This formula was discovered by Newton.

178. Suppose we have two expressions arranged in ascending powers of x, such as

$$1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \ldots\ldots\ldots \quad (1),$$

and $$1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \ldots\ldots\ldots\ldots \quad (2).$$

The product of these two expressions will be a series in ascending powers of x; denote it by

$$1 + Ax + Bx^2 + Cx^3 + Dx^4 + \ldots\ldots;$$

then it is clear that $A, B, C, \ldots\ldots$ are functions of m and n, and therefore the actual values of $A, B, C, \ldots\ldots$ in any particular case will depend upon the values of m and n in that case. But the way in which the coefficients of the powers of x in (1) and (2) combine to give $A, B, C, \ldots\ldots$ is quite independent of m and n; in other words, *whatever values* m *and* n *may have,* A, B, C, ... *preserve the same invariable form.* If therefore we can determine the form of $A, B, C, \ldots\ldots$ for any value of m and n, we conclude that $A, B, C, \ldots\ldots$ will have the same form *for all values* of m and n.

The principle here explained is often referred to as an example of "the permanence of equivalent forms;" in the present case we have only to recognise the fact that *in any algebraical product* the *form* of the result will be the same whether the quantities involved are whole numbers, or fractions; positive, or negative.

We shall make use of this principle in the general proof of the Binomial Theorem for any index. The proof which we give is due to Euler.

179. *To prove the Binomial Theorem when the index is a positive fraction.*

Whatever be the value of m, *positive or negative, integral or fractional,* let the symbol $f(m)$ stand for the series

$$1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \ldots;$$

then $f(n)$ will stand for the series

$$1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \ldots.$$

If we multiply these two series together the product will be another series in ascending powers of x, whose *coefficients will be unaltered in form whatever* m *and* n *may be.*

To determine this *invariable form of the product* we may give to m and n any values that are most convenient; for this purpose suppose that m and n are positive integers. In this case $f(m)$ is the expanded form of $(1+x)^m$, and $f(n)$ is the expanded form of $(1+x)^n$; and therefore

$$f(m) \times f(n) = (1+x)^m \times (1+x)^n = (1+x)^{m+n},$$

but when m and n are positive integers the expansion of $(1+x)^{m+n}$ is

$$1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 \cdot 2} x^2 + \ldots\ldots.$$

This then is the *form* of the product of $f(m) \times f(n)$ *in all cases*, whatever the values of m and n may be; and in agreement with our previous notation it may be denoted by $f(m+n)$; therefore *for all values of* m *and* n

$$f(m) \times f(n) = f(m+n).$$

Also
$$f(m) \times f(n) \times f(p) = f(m+n) \times f(p)$$
$$= f(m+n+p), \text{ similarly.}$$

Proceeding in this way we may shew that

$$f(m) \times f(n) \times f(p) \ldots \text{to } k \text{ factors} = f(m+n+p+\ldots \text{to } k \text{ terms}).$$

Let each of these quantities m, n, p, ... be equal to $\frac{h}{k}$, where h and k are positive integers;

$$\therefore \quad \left\{f\left(\frac{h}{k}\right)\right\}^k = f(h);$$

but since h is a positive integer, $f(h) = (1+x)^h$;

$$\therefore \quad (1+x)^h = \left\{f\left(\frac{h}{k}\right)\right\}^k;$$

$$\therefore \quad (1+x)^{\frac{h}{k}} = f\left(\frac{h}{k}\right);$$

but $f\left(\frac{h}{k}\right)$ stands for the series

$$1 + \frac{h}{k}x + \frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1 \cdot 2} x^2 + \ldots\ldots;$$

$$\therefore \quad (1+x)^{\frac{h}{k}} = 1 + \frac{h}{k}x + \frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1 \cdot 2} x^2 + \ldots\ldots,$$

which proves the Binomial Theorem for any positive fractional index.

180. *To prove the Binomial Theorem when the index is any negative quantity.*

It has been proved that

$$f(m) \times f(n) = f(m+n)$$

for all values of m and n. Replacing m by $-n$ (where n is positive), we have

$$\begin{aligned} f(-n) \times f(n) &= f(-n+n) \\ &= f(0) \\ &= 1, \end{aligned}$$

since all terms of the series except the first vanish;

$$\therefore \quad \frac{1}{f(n)} = f(-n);$$

but $f(n) = (1+x)^n$, for *any positive* value of n;

$$\therefore \quad \frac{1}{(1+x)^n} = f(-n),$$

or
$$(1+x)^{-n} = f(-n).$$

But $f(-n)$ stands for the series

$$1 + (-n)x + \frac{(-n)(-n-1)}{1 \cdot 2} x^2 + \ldots\ldots;$$

$$\therefore \quad (1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{1 \cdot 2} x^2 + \ldots\ldots;$$

which proves the Binomial Theorem for any negative index. Hence the theorem is completely established.

181. The proof contained in the two preceding articles may not appear wholly satisfactory, and will probably present some difficulties to the student. There is only one point to which we shall now refer.

In the expression for $f(m)$ the number of terms is finite when m is a positive integer, and unlimited in all other cases. See Art. 182. It is therefore necessary to enquire in what sense we are to regard the statement that $f(m) \times f(n) = f(m+n)$. It will be seen in Chapter XXI., that when $x < 1$, each of the series $f(m)$, $f(n)$, $f(m+n)$ is *convergent*, and $f(m+n)$ is the true arithmetical equivalent of $f(m) \times f(n)$. But when $x > 1$, all these series are *divergent*, and we can only assert that if we multiply the series denoted by $f(m)$ by the series denoted by $f(n)$, the first r terms of the product will agree with the first r terms of $f(m+n)$, whatever finite value r may have. [See Art. 308.]

Example 1. Expand $(1-x)^{\frac{3}{2}}$ to four terms.

$$(1-x)^{\frac{3}{2}} = 1 + \frac{3}{2}(-x) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1 \cdot 2}(-x)^2 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1 \cdot 2 \cdot 3}(-x)^3 + \ldots\ldots$$

$$= 1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \ldots\ldots$$

Example 2. Expand $(2+3x)^{-4}$ to four terms.

$$(2+3x)^{-4} = 2^{-4}\left(1 + \frac{3x}{2}\right)^{-4}$$

$$= \frac{1}{2^4}\left[1 + (-4)\left(\frac{3x}{2}\right) + \frac{(-4)(-5)}{1 \cdot 2}\left(\frac{3x}{2}\right)^2 + \frac{(-4)(-5)(-6)}{1 \cdot 2 \cdot 3}\left(\frac{3x}{2}\right)^3 + \ldots\ldots\right]$$

$$= \frac{1}{16}\left(1 - 6x + \frac{45}{2}x^2 - \frac{135}{2}x^3 + \ldots\ldots\right).$$

182. In finding the general term we must now use the formula

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}}x^r$$

written in full; for the symbol ${}^nC_r$ can no longer be employed when n is fractional or negative.

Also the coefficient of the general term can never vanish unless one of the factors of its numerator is zero; the series will therefore stop at the r^{th} term, when $n-r+1$ is zero; that is, when $r = n+1$; but since r is a positive integer this equality can never hold except when the index n is positive and integral. Thus the expansion by the Binomial Theorem extends to $n+1$ terms when n is a positive integer, and to an infinite number of terms in all other cases.

Example 1. Find the general term in the expansion of $(1+x)^{\frac{1}{2}}$.

$$\text{The } (r+1)^{\text{th}} \text{ term} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\ldots\left(\frac{1}{2}-r+1\right)}{\lfloor r} x^r$$

$$= \frac{1(-1)(-3)(-5)\ldots\ldots(-2r+3)}{2^r \lfloor r} x^r.$$

The number of factors in the numerator is r, and $r-1$ of these are negative; therefore, by taking -1 out of each of these negative factors, we may write the above expression

$$(-1)^{r-1} \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2r-3)}{2^r \lfloor r} x^r.$$

Example 2. Find the general term in the expansion of $(1-nx)^{\frac{1}{n}}$.

$$\text{The } (r+1)^{\text{th}} \text{ term} = \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\ldots\ldots\left(\frac{1}{n}-r+1\right)}{\lfloor r} (-nx)^r$$

$$= \frac{1(1-n)(1-2n)\ldots\ldots(1-\overline{r-1}\cdot n)}{n^r \lfloor r} (-1)^r n^r x^r$$

$$= (-1)^r \frac{1(1-n)(1-2n)\ldots\ldots(1-\overline{r-1}\cdot n)}{\lfloor r} x^r$$

$$= (-1)^r (-1)^{r-1} \frac{(n-1)(2n-1)\ldots\ldots(\overline{r-1}\cdot n-1)}{\lfloor r} x^r$$

$$= -\frac{(n-1)(2n-1)\ldots\ldots(\overline{r-1}\cdot n-1)}{\lfloor r} x^r,$$

since $$(-1)^r(-1)^{r-1} = (-1)^{2r-1} = -1.$$

Example 3. Find the general term in the expansion of $(1-x)^{-3}$.

$$\text{The } (r+1)^{\text{th}} \text{ term} = \frac{(-3)(-4)(-5)\ldots\ldots(-3-r+1)}{\lfloor r} (-x)^r$$

$$= (-1)^r \frac{3 \cdot 4 \cdot 5 \ldots\ldots (r+2)}{\lfloor r} (-1)^r x^r$$

$$= (-1)^{2r} \frac{3 \cdot 4 \cdot 5 \ldots\ldots (r+2)}{1 \cdot 2 \cdot 3 \ldots\ldots r} x^r$$

$$= \frac{(r+1)(r+2)}{1 \cdot 2} x^r,$$

by removing like factors from the numerator and denominator.

EXAMPLES. XIV. a.

Expand to 4 terms the following expressions:

1. $(1+x)^{\frac{1}{2}}$.
2. $(1+x)^{\frac{3}{2}}$.
3. $(1-x)^{\frac{2}{5}}$.
4. $(1+x^2)^{-2}$.
5. $(1-3x)^{\frac{1}{3}}$.
6. $(1-3x)^{-\frac{1}{3}}$.
7. $(1+2x)^{-\frac{1}{2}}$.
8. $\left(1+\frac{x}{3}\right)^{-3}$.
9. $\left(1+\frac{2x}{3}\right)^{\frac{3}{2}}$.
10. $\left(1+\frac{1}{2}a\right)^{-4}$.
11. $(2+x)^{-3}$.
12. $(9+2x)^{\frac{1}{2}}$.
13. $(8+12a)^{\frac{2}{3}}$.
14. $(9-6x)^{-\frac{3}{2}}$.
15. $(4a-8x)^{-\frac{1}{2}}$.

Write down and simplify:

16. The 8^{th} term of $(1+2x)^{-\frac{1}{2}}$.
17. The 11^{th} term of $(1-2x^3)^{\frac{11}{2}}$.
18. The 10^{th} term of $(1+3a^2)^{\frac{16}{3}}$.
19. The 5^{th} term of $(3a-2b)^{-1}$.
20. The $(r+1)^{\text{th}}$ term of $(1-x)^{-2}$.
21. The $(r+1)^{\text{th}}$ term of $(1-x)^{-4}$.
22. The $(r+1)^{\text{th}}$ term of $(1+x)^{\frac{1}{2}}$.
23. The $(r+1)^{\text{th}}$ term of $(1+x)^{\frac{11}{3}}$.
24. The 14^{th} term of $(2^{10}-2^7x)^{\frac{13}{2}}$.
25. The 7^{th} term of $(3^8+6^4x)^{\frac{11}{4}}$.

183. If we expand $(1-x)^{-2}$ by the Binomial Theorem, we obtain

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots\dots;$$

but, by referring to Art. 60, we see that this result is only true when x is less than 1. This leads us to enquire whether we are always justified in assuming the truth of the statement

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \dots\dots,$$

and, if not, under what conditions the expansion of $(1+x)^n$ may be used as its true equivalent.

Suppose, for instance, that $n=-1$; then we have

$$(1-x)^{-1}=1+x+x^2+x^3+x^4+\ldots\ldots\ldots\ldots \quad (1);$$

in this equation put $x=2$; we then obtain

$$(-1)^{-1}=1+2+2^2+2^3+2^4+\ldots\ldots.$$

This contradictory result is sufficient to shew that we cannot take

$$1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\ldots\ldots$$

as the true arithmetical equivalent of $(1+x)^n$ in all cases.

Now from the formula for the sum of a geometrical progression, we know that the sum of the first r terms of the series (1)

$$=\frac{1-x^r}{1-x}$$

$$=\frac{1}{1-x}-\frac{x^r}{1-x};$$

and, when x is numerically less than 1, by taking r sufficiently large we can make $\frac{x^r}{1-x}$ as small as we please; that is, by taking a sufficient number of terms the sum can be made to differ as little as we please from $\frac{1}{1-x}$. But when x is numerically greater than 1, the value of $\frac{x^r}{1-x}$ increases with r, and therefore no such approximation to the value of $\frac{1}{1-x}$ is obtained by taking any number of terms of the series

$$1+x+x^2+x^3+\ldots\ldots.$$

It will be seen in the chapter on Convergency and Divergency of Series that the expansion by the Binomial Theorem of $(1+x)^n$ in ascending powers of x is always arithmetically intelligible when x is less than 1.

But if x is greater than 1, then since the general term of the series

$$1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\ldots\ldots$$

contains x^r, it can be made greater than any finite quantity by taking r sufficiently large; in which case there is no limit to the value of the above series; and therefore the expansion of $(1+x)^n$ as an infinite series in ascending powers of x has no meaning arithmetically intelligible when x is greater than 1.

184. We may remark that we can always expand $(x+y)^n$ by the Binomial Theorem; for we may write the expression in either of the two following forms:

$$x^n\left(1+\frac{y}{x}\right)^n, \qquad y^n\left(1+\frac{x}{y}\right)^n;$$

and we obtain the expansion from the first or second of these according as x is greater or less than y.

185. *To find in its simplest form the general term in the expansion of* $(1-x)^{-n}$.

The $(r+1)^{\text{th}}$ term

$$=\frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{\lfloor r}(-x)^r$$

$$=(-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}(-1)^r x^r$$

$$=(-1)^{2r}\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}x^r$$

$$=\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}x^r.$$

From this it appears that every term in the expansion of $(1-x)^{-n}$ is positive.

Although the general term in the expansion of any binomial may always be found as explained in Art. 182, it will be found more expeditious in practice to use the above form of the general term in all cases where the index is negative, retaining the form

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}x^r$$

only in the case of positive indices.

Example. Find the general term in the expansion of $\dfrac{1}{\sqrt[3]{1-3x}}$.

$$\frac{1}{\sqrt[3]{1-3x}} = (1-3x)^{-\frac{1}{3}}.$$

The $(r+1)^{\text{th}}$ term

$$= \frac{\frac{1}{3}\left(\frac{1}{3}+1\right)\left(\frac{1}{3}+2\right)\ldots\ldots\left(\frac{1}{3}+r-1\right)}{\underline{|r}}(3x)^r$$

$$= \frac{1 \,.\, 4 \,.\, 7 \ldots\ldots (3r-2)}{3^r \underline{|r}} 3^r x^r$$

$$= \frac{1 \,.\, 4 \,.\, 7 \ldots\ldots (3r-2)}{\underline{|r}} x^r.$$

If the given expression had been $(1+3x)^{-\frac{1}{3}}$ we should have used the same formula for the general term, replacing $3x$ by $-3x$.

186. The following expansions should be remembered:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots + x^r + \ldots\ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (r+1)\,x^r + \ldots\ldots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{1 \,.\, 2} x^r + \ldots\ldots$$

187. The general investigation of the greatest term in the expansion of $(1+x)^n$, when n is unrestricted in value, will be found in Art. 189; but the student will have no difficulty in applying to any numerical example the method explained in Art. 172.

Example. Find the greatest term in the expansion of $(1+x)^{-n}$ when $x = \dfrac{2}{3}$, and $n = 20$.

We have
$$T_{r+1} = \frac{n+r-1}{r} \,.\, x \times T_r, \textit{ numerically},$$
$$= \frac{19+r}{r} \times \frac{2}{3} \times T_r;$$
$$\therefore\ T_{r+1} > T_r,$$
so long as
$$\frac{2(19+r)}{3r} > 1;$$
that is,
$$38 > r.$$

Hence for all values of r up to 37, we have $T_{r+1} > T_r$; but if $r = 38$, then $T_{r+1} = T_r$, and these are the greatest terms. Thus the 38th and 39th terms are equal numerically and greater than any other term.

188. Some useful applications of the Binomial Theorem are explained in the following examples.

Example 1. Find the first three terms in the expansion of

$$(1+3x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{3}}.$$

Expanding the two binomials as far as the term containing x^2, we have

$$\left(1+\frac{3}{2}x-\frac{9}{8}x^2-\ldots\right)\left(1+\frac{2}{3}x+\frac{8}{9}x^2+\ldots\right)$$

$$=1+x\left(\frac{3}{2}+\frac{2}{3}\right)+x^2\left(\frac{8}{9}+\frac{3}{2}\cdot\frac{2}{3}-\frac{9}{8}\right)\ldots\ldots$$

$$=1+\frac{13}{6}x+\frac{55}{72}x^2.$$

If in this Example $x=\cdot002$, so that $x^2=\cdot000004$, we see that the third term is a decimal fraction beginning with 5 ciphers. If therefore we were required to find the numerical value of the given expression correct to 5 places of decimals it would be sufficient to substitute $\cdot002$ for x in $1+\frac{13}{6}x$, neglecting the term involving x^2.

Example 2. When x is so small that its square and higher powers may be neglected, find the value of

$$\frac{\left(1+\frac{2}{3}x\right)^{-5}+\sqrt{4+2x}}{\sqrt{(4+x)^3}}.$$

Since x^2 and the higher powers may be neglected, it will be sufficient to retain the first two terms in the expansion of each binomial. Therefore

the expression $$=\frac{\left(1+\frac{2}{3}x\right)^{-5}+2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}}{8\left(1+\frac{x}{4}\right)^{\frac{3}{2}}}$$

$$=\frac{\left(1-\frac{10}{3}x\right)+2\left(1+\frac{1}{4}x\right)}{8\left(1+\frac{3}{8}x\right)}$$

$$=\frac{1}{8}\left(3-\frac{17}{6}x\right)\left(1+\frac{3}{8}x\right)^{-1}$$

$$=\frac{1}{8}\left(3-\frac{17}{6}x\right)\left(1-\frac{3}{8}x\right)$$

$$=\frac{1}{8}\left(3-\frac{95}{24}x\right),$$

the term involving x^2 being neglected.

Example 3. Find the value of $\frac{1}{\sqrt{47}}$ to four places of decimals.

$$\frac{1}{\sqrt{47}}=(47)^{-\frac{1}{2}}=(7^2-2)^{-\frac{1}{2}}=\frac{1}{7}\left(1-\frac{2}{7^2}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{7}\left(1+\frac{1}{7^2}+\frac{3}{2}\cdot\frac{1}{7^4}+\frac{5}{2}\cdot\frac{1}{7^6}+\dots\right)$$

$$=\frac{1}{7}+\frac{1}{7^3}+\frac{3}{2}\cdot\frac{1}{7^5}+\frac{5}{2}\cdot\frac{1}{7^7}+\dots$$

To obtain the values of the several terms we proceed as follows:

$$\begin{array}{rl}
7\,)\,1 & \\
7\,)\,\underline{\cdot142857} \dots\dots\dots\dots & =\frac{1}{7}, \\
7\,)\,\underline{\cdot020408} & \\
7\,)\,\underline{\cdot002915} \dots\dots\dots\dots & =\frac{1}{7^3}, \\
7\,)\,\underline{\cdot000416} & \\
\cdot000059 \dots\dots\dots\dots & =\frac{1}{7^5};
\end{array}$$

and we can see that the term $\frac{5}{2}\cdot\frac{1}{7^7}$ is a decimal fraction beginning with 5 ciphers.

$$\therefore \frac{1}{\sqrt{47}}=\cdot142857+\cdot002915+\cdot000088$$

$$=\cdot14586,$$

and this result is correct to at least four places of decimals.

Example 4. Find the cube root of 126 to 5 places of decimals.

$$(126)^{\frac{1}{3}}=(5^3+1)^{\frac{1}{3}}$$

$$=5\left(1+\frac{1}{5^3}\right)^{\frac{1}{3}}$$

$$=5\left(1+\frac{1}{3}\cdot\frac{1}{5^3}-\frac{1}{9}\cdot\frac{1}{5^6}+\frac{5}{81}\cdot\frac{1}{5^9}-\dots\right)$$

$$=5+\frac{1}{3}\cdot\frac{1}{5^2}-\frac{1}{9}\cdot\frac{1}{5^5}+\frac{1}{81}\cdot\frac{1}{5^7}-\dots$$

$$=5+\frac{1}{3}\cdot\frac{2^2}{10^2}-\frac{1}{9}\cdot\frac{2^5}{10^5}+\frac{1}{81}\cdot\frac{2^7}{10^7}-\dots$$

$$=5+\frac{\cdot04}{3}-\frac{\cdot00032}{9}+\frac{\cdot0000128}{81}-\dots$$

$$=5+\cdot013333\dots-\cdot000035\dots+\dots$$

$$=5\cdot01329, \text{ to five places of decimals.}$$

EXAMPLES. XIV. b.

Find the $(r+1)^{\text{th}}$ term in each of the following expansions:

1. $(1+x)^{-\frac{1}{2}}$. 2. $(1-x)^{-5}$. 3. $(1+3x)^{\frac{1}{3}}$.

4. $(1+x)^{-\frac{2}{3}}$. 5. $(1+x^2)^{-3}$. 6. $(1-2x)^{-\frac{3}{2}}$.

7. $(a+bx)^{-1}$. 8. $(2-x)^{-2}$. 9. $\sqrt[3]{(a^3-x^3)^2}$.

10. $\dfrac{1}{\sqrt{1+2x}}$. 11. $\dfrac{1}{\sqrt[3]{(1-3x)^2}}$. 12. $\dfrac{1}{\sqrt[n]{a^n-nx}}$.

Find the greatest term in each of the following expansions:

13. $(1+x)^{-7}$ when $x=\dfrac{4}{15}$.

14. $(1+x)^{\frac{21}{2}}$ when $x=\dfrac{2}{3}$.

15. $(1-7x)^{-\frac{11}{4}}$ when $x=\dfrac{1}{8}$.

16. $(2x+5y)^{12}$ when $x=8$ and $y=3$.

17. $(5-4x)^{-7}$ when $x=\dfrac{1}{2}$.

18. $(3x^2+4y^3)^{-n}$ when $x=9$, $y=2$, $n=15$.

Find to five places of decimals the value of

19. $\sqrt{98}$. 20. $\sqrt[3]{998}$. 21. $\sqrt[3]{1003}$. 22. $\sqrt[4]{2400}$.

23. $\dfrac{1}{\sqrt[3]{128}}$. 24. $(1\frac{1}{250})^{\frac{1}{3}}$. 25. $(630)^{-\frac{3}{4}}$. 26. $\sqrt[5]{3128}$.

If x be so small that its square and higher powers may be neglected, find the value of

27. $(1-7x)^{\frac{1}{3}}(1+2x)^{-\frac{3}{4}}$. 28. $\sqrt{4-x}\,.\left(3-\dfrac{x}{2}\right)^{-1}$.

29. $\dfrac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}}$. 30. $\dfrac{\left(1+\frac{2}{3}x\right)^{-5}\times(4+3x)^{\frac{1}{2}}}{(4+x)^{\frac{3}{2}}}$.

31. $\dfrac{\sqrt[4]{1-\frac{3}{5}x}+\left(1+\frac{5}{6}x\right)^{-6}}{\sqrt[3]{1+2x}+\sqrt[5]{1-\frac{x}{2}}}$.   32. $\dfrac{\sqrt[3]{8+3x}-\sqrt[5]{1-x}}{(1+5x)^{\frac{3}{5}}+\left(4+\frac{x}{2}\right)^{\frac{1}{2}}}$.

33. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\dfrac{|\underline{2r}}{(|\underline{r})^2}$.

34. Prove that $(1+x)^n=2^n\left\{1-n\dfrac{1-x}{1+x}+\dfrac{n(n+1)}{1\,.\,2}\left(\dfrac{1-x}{1+x}\right)^2\ldots\ldots\right\}$.

35. Find the first three terms in the expansion of

$$\frac{1}{(1+x)^2\sqrt{1+4x}}.$$

36. Find the first three terms in the expansion of

$$\frac{(1+x)^{\frac{3}{4}}+\sqrt{1+5x}}{(1-x)^2}.$$

37. Shew that the n^{th} coefficient in the expansion of $(1-x)^{-n}$ is double of the $(n-1)^{\text{th}}$.

189. *To find the numerically greatest term in the expansion of* $(1+\mathrm{x})^{\mathrm{n}}$, *for any rational value of* n.

Since we are only concerned with the *numerical* value of the greatest term, we shall consider x throughout as positive.

Case I. Let n be a positive integer.

The $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\dfrac{n-r+1}{r}\,.\,x$; that is, by $\left(\dfrac{n+1}{r}-1\right)x$; and therefore the terms continue to increase so long as

$$\left(\frac{n+1}{r}-1\right)x>1;$$

that is,
$$\frac{(n+1)x}{r}>1+x,$$

or
$$\frac{(n+1)x}{1+x}>r.$$

If $\dfrac{(n+1)x}{1+x}$ be an integer, denote it by p; then if $r=p$, the multiplying factor is 1, and the $(p+1)^{\text{th}}$ term is equal to the p^{th}, and these are greater than any other term.

If $\dfrac{(n+1)x}{1+x}$ be not an integer, denote its integral part by q; then the greatest value of r is q, and the $(q+1)^{\text{th}}$ term is the greatest.

Case II. Let n be a positive fraction.

As before, the $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\left(\dfrac{n+1}{r}-1\right)x$.

(1) If x be greater than unity, by increasing r the above multiplier can be made as near as we please to $-x$; so that after a certain term each term is nearly x times the preceding term numerically, and thus the terms increase continually, and there is no greatest term.

(2) If x be less than unity we see that the multiplying factor continues positive, and decreases until $r > n+1$, and from this point it becomes negative but always remains less than 1 numerically; therefore there will be a greatest term.

As before, the multiplying factor will be greater than 1 so long as
$$\frac{(n+1)x}{1+x} > r.$$

If $\dfrac{(n+1)x}{1+x}$ be an integer, denote it by p; then, as in Case I., the $(p+1)^{\text{th}}$ term is equal to the p^{th}, and these are greater than any other term.

If $\dfrac{(n+1)x}{1+x}$ be not an integer, let q be its integral part; then the $(q+1)^{\text{th}}$ term is the greatest.

Case III. Let n be negative.

Let $n=-m$, so that m is positive; then the numerical value of the multiplying factor is $\dfrac{m+r-1}{r}\,.\,x$; that is
$$\left(\frac{m-1}{r}+1\right)x.$$

(1) If x be greater than unity we may shew, as in Case II., that there is no greatest term.

(2) If x be less than unity, the multiplying factor will be greater than 1, so long as

$$\left(\frac{m-1}{r}+1\right)x > 1;$$

that is,
$$\frac{(m-1)x}{r} > 1-x,$$

or
$$\frac{(m-1)x}{1-x} > r.$$

If $\dfrac{(m-1)x}{1-x}$ be a positive integer, denote it by p; then the $(p+1)^{\text{th}}$ term is equal to the p^{th} term, and these are greater than any other term.

If $\dfrac{(m-1)x}{1-x}$ be positive but not an integer, let q be its integral part; then the $(q+1)^{\text{th}}$ term is the greatest.

If $\dfrac{(m-1)x}{1-x}$ be negative, then m is less than unity; and by writing the multiplying factor in the form $\left(1-\dfrac{1-m}{r}\right)x$, we see that it is always less than 1: hence each term is less than the preceding, and consequently the first term is the greatest.

190. *To find the number of homogeneous products of* r *dimensions that can be formed out of the* n *letters* a, b, c, ... *and their powers.*

By division, or by the Binomial Theorem, we have

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \ldots\ldots,$$

$$\frac{1}{1-bx} = 1 + bx + b^2x^2 + b^3x^3 + \ldots\ldots,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \ldots\ldots,$$

....

Hence, by multiplication,

$$\frac{1}{1-ax}\cdot\frac{1}{1-bx}\cdot\frac{1}{1-cx}\cdot\ldots\ldots$$

$$= (1 + ax + a^2x^2 + \ldots)(1 + bx + b^2x^2 + \ldots)(1 + cx + c^2x^2 + \ldots)\ldots$$

$$= 1 + x(a + b + c + \ldots) + x^2(a^2 + ab + ac + b^2 + bc + c^2 + \ldots) + \ldots$$

$$= 1 + S_1x + S_2x^2 + S_3x^3 + \ldots\ldots \text{ suppose};$$

where S_1, S_2, S_3, ... are the *sums* of the homogeneous products of *one, two, three,* ...... dimensions that can be formed of a, b, c, ... and their powers.

To obtain the *number* of these products, put a, b, c, ... each equal to 1; each term in S_1, S_2, S_3, ... now becomes 1, and the values of S_1, S_2, S_3, ... so obtained give the *number* of the homogeneous products of *one*, *two*, *three*, ... dimensions.

Also $$\frac{1}{1-ax}\cdot\frac{1}{1-bx}\cdot\frac{1}{1-cx}\ldots\ldots$$

becomes $$\frac{1}{(1-x)^n} \text{ or } (1-x)^{-n}.$$

Hence S_r = coefficient of x^r in the expansion of $(1-x)^{-n}$

$$= \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{\lfloor r}$$

$$= \frac{\lfloor n+r-1}{\lfloor r\,\lfloor n-1}.$$

191. *To find the number of terms in the expansion of any multinomial when the index is a positive integer.*

In the expansion of

$$(a_1 + a_2 + a_3 + \ldots\ldots + a_r)^n,$$

every term is of n dimensions; therefore the number of terms is the same as the number of homogeneous products of n dimensions that can be formed out of the r quantities a_1, a_2, ... a_r, and their powers; and therefore by the preceding article is equal to

$$\frac{\lfloor r+n-1}{\lfloor n\,\lfloor r-1}.$$

192. From the result of Art. 190 we may deduce a theorem relating to the number of combinations of n things.

Consider n letters $a, b, c, d, \ldots\ldots$; then if we were to write down all the homogeneous products of r dimensions which can be formed of these letters and their powers, every such product would represent one of the combinations, r at a time, of the n letters, when any one of the letters might occur once, twice, thrice, ... up to r times.

Therefore the number of combinations of n things r at a time when repetitions are allowed is equal to the number of homogeneous products of r dimensions which can be formed out of n letters, and therefore equal to $\dfrac{|\underline{n+r-1}}{|\underline{r}\ |\underline{n-1}}$, or ${}^{n+r-1}C_r$.

That is, the number of combinations of n things r at a time when repetitions are allowed is equal to the number of combinations of $n+r-1$ things r at a time when repetitions are excluded.

193. We shall conclude this chapter with a few miscellaneous examples.

Example 1. Find the coefficient of x^r in the expansion of $\dfrac{(1-2x)^2}{(1+x)^3}$.

The expression $=(1-4x+4x^2)(1+p_1x+p_2x^2+\ldots+p_rx^r+\ldots)$ suppose.

The coefficient of x^r will be obtained by multiplying p_r, p_{r-1}, p_{r-2} by 1, -4, 4 respectively, and adding the results; hence

$$\text{the required coefficient} = p_r - 4p_{r-1} + 4p_{r-2}.$$

But
$$p_r = (-1)^r \frac{(r+1)(r+2)}{2}. \qquad \text{[Ex. 3, Art. 182.]}$$

Hence the required coefficient

$$= (-1)^r \frac{(r+1)(r+2)}{2} - 4(-1)^{r-1} \frac{r(r+1)}{2} + 4(-1)^{r-2} \frac{(r-1)r}{2}$$

$$= \frac{(-1)^r}{2} [(r+1)(r+2) + 4r(r+1) + 4r(r-1)]$$

$$= \frac{(-1)^r}{2} (9r^2 + 3r + 2).$$

Example 2. Find the value of the series

$$2+\frac{5}{\lfloor 2 \cdot 3}+\frac{5 \cdot 7}{\lfloor 3 \cdot 3^2}+\frac{5 \cdot 7 \cdot 9}{\lfloor 4 \cdot 3^3}+\dots$$

The expression $=2+\dfrac{3 \cdot 5}{\lfloor 2} \cdot \dfrac{1}{3^2}+\dfrac{3 \cdot 5 \cdot 7}{\lfloor 3} \cdot \dfrac{1}{3^3}+\dfrac{3 \cdot 5 \cdot 7 \cdot 9}{\lfloor 4} \cdot \dfrac{1}{3^4}+\dots$

$$=2+\frac{\frac{3}{2} \cdot \frac{5}{2}}{\lfloor 2} \cdot \frac{2^2}{3^2}+\frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}}{\lfloor 3} \cdot \frac{2^3}{3^3}+\frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2}}{\lfloor 4} \cdot \frac{2^4}{3^4}+\dots$$

$$=1+\frac{\frac{3}{2}}{1} \cdot \frac{2}{3}+\frac{\frac{3}{2} \cdot \frac{5}{2}}{\lfloor 2} \cdot \left(\frac{2}{3}\right)^2+\frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}}{\lfloor 3} \cdot \left(\frac{2}{3}\right)^3+\dots$$

$$=\left(1-\frac{2}{3}\right)^{-\frac{3}{2}}=\left(\frac{1}{3}\right)^{-\frac{3}{2}}$$

$$=3^{\frac{3}{2}}=3\sqrt{3}.$$

Example 3. If n is any positive integer, shew that the integral part of $(3+\sqrt{7})^n$ is an odd number.

Suppose I to denote the integral and f the fractional part of $(3+\sqrt{7})^n$. Then $\quad I+f=3^n+C_1 3^{n-1}\sqrt{7}+C_2 3^{n-2} \cdot 7+C_3 3^{n-3}(\sqrt{7})^3+\dots\dots\dots\dots(1).$

Now $3-\sqrt{7}$ is positive and less than 1, therefore $(3-\sqrt{7})^n$ is a proper fraction; denote it by f';

$$\therefore\ f'=3^n-C_1 3^{n-1}\sqrt{7}+C_2 3^{n-2} \cdot 7+C_3 3^{n-3}(\sqrt{7})^3+\dots\dots\dots\dots(2).$$

Add together (1) and (2); the irrational terms disappear, and we have

$$I+f+f'=2(3^n+C_2 3^{n-2} \cdot 7+\dots)$$
$$=\text{an even integer.}$$

But since f and f' are proper fractions their sum must be 1;

$$\therefore\ I=\text{an odd integer.}$$

EXAMPLES. XIV. c.

Find the coefficient of

1. x^{100} in the expansion of $\dfrac{3-5x}{(1-x)^2}$.

2. a^{12} in the expansion of $\dfrac{4+2a-a^2}{(1+a)^3}$.

3. x^n in the expansion of $\dfrac{3x^2-2}{x+x^2}$.

4. Find the coefficient of x^n in the expansion of $\dfrac{2+x+x^2}{(1+x)^3}$.

5. Prove that

$$1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2^2}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{2^3}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{1}{2^4}-\ldots=\sqrt{\frac{2}{3}}.$$

6. Prove that

$$\sqrt{8}=1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots\ldots$$

7. Prove that

$$1+\frac{2n}{3}+\frac{2n(2n+2)}{3.6}+\frac{2n(2n+2)(2n+4)}{3.6.9}+\ldots\ldots$$

$$=2^n\left\{1+\frac{n}{3}+\frac{n(n+1)}{3.6}+\frac{n(n+1)(n+2)}{3.6.9}+\ldots\ldots\right\}.$$

8. Prove that

$$7^n\left\{1+\frac{n}{7}+\frac{n(n-1)}{7.14}+\frac{n(n-1)(n-2)}{7.14.21}+\ldots\ldots\right\}$$

$$=4^n\left\{1+\frac{n}{2}+\frac{n(n+1)}{2.4}+\frac{n(n+1)(n+2)}{2.4.6}+\ldots\ldots\right\}.$$

9. Prove that approximately, when x is very small,

$$\frac{3\left(x+\frac{4}{9}\right)^{\frac{1}{2}}\left(1-\frac{3}{4}x^2\right)^{\frac{1}{3}}}{2\left(1+\frac{9}{16}x\right)^2}=1-\frac{307}{256}x^2.$$

10. Shew that the integral part of $(5+2\sqrt{6})^n$ is odd, if n be a positive integer.

11. Shew that the integral part of $(8+3\sqrt{7})^n$ is odd, if n be a positive integer.

12. Find the coefficient of x^n in the expansion of

$$(1-2x+3x^2-4x^3+\ldots\ldots)^{-n}.$$

13. Shew that the middle term of $\left(x+\dfrac{1}{x}\right)^{4n}$ is equal to the coefficient of x^n in the expansion of $(1-4x)^{-\left(n+\frac{1}{2}\right)}$.

14. Prove that the expansion of $(1-x^3)^n$ may be put into the form

$$(1-x)^{3n}+3nx(1-x)^{3n-2}+\frac{3n(3n-3)}{1.2}x^2(1-x)^{3n-4}+\ldots\ldots$$

15. Prove that the coefficient of x^n in the expansion $\dfrac{1}{1+x+x^2}$ is $1,\ 0,\ -1$ according as n is of the form $3m$, $3m-1$, or $3m+1$.

16. In the expansion of $(a+b+c)^8$ find (1) the number of terms, (2) the sum of the coefficients of the terms.

17. Prove that if n be an even integer,

$$\frac{1}{\underline{|1}\,\underline{|n-1}}+\frac{1}{\underline{|3}\,\underline{|n-3}}+\frac{1}{\underline{|5}\,\underline{|n-5}}+\ldots\ldots+\frac{1}{\underline{|n-1}\,\underline{|1}}=\frac{2^{n-1}}{\underline{|n}}.$$

18. If $c_0,\ c_1,\ c_2,\ \ldots\ldots\ c_n$ are the coefficients in the expansion of $(1+x)^n$, when n is a positive integer, prove that

(1) $\quad c_0-c_1+c_2-c_3+\ldots\ldots+(-1)^r c_r=(-1)^r\dfrac{\underline{|n-1}}{\underline{|r}\,\underline{|n-r-1}}.$

(2) $\quad c_0-2c_1+3c_2-4c_3+\ldots\ldots+(-1)^n(n+1)c_n=0.$

(3) $\quad c_0^2-c_1^2+c_2^2-c_3^2+\ldots\ldots+(-1)^n c_n^2=0$, or $(-1)^{\frac{n}{2}}c_{\frac{n}{2}}$,

according as n is odd or even.

19. If s_n denote the sum of the first n natural numbers, prove that

(1) $\quad (1-x)^{-3}=s_1+s_2x+s_3x^2+\ldots\ldots+s_nx^{n-1}+\ldots$

(2) $\quad 2(s_1s_{2n}+s_2s_{2n-1}+\ldots\ldots+s_ns_{n+1})=\dfrac{\underline{|2n+4}}{\underline{|5}\,\underline{|2n-1}}.$

20. If $q_n=\dfrac{1\,.\,3\,.\,5\,.\,7\ldots\ldots(2n-1)}{2\,.\,4\,.\,6\,.\,8\ldots\ldots2n}$, prove that

(1) $\quad q_{2n+1}+q_1q_{2n}+q_2q_{2n-1}+\ldots\ldots+q_{n-1}q_{n+2}+q_nq_{n+1}=\dfrac{1}{2}.$

(2) $\quad 2\{q_{2n}-q_1q_{2n-1}+q_2q_{2n-2}+\ldots\ldots+(-1)^{n-1}q_{n-1}q_{n+1}\}$
$$=q_n+(-1)^{n-1}q_n^2.$$

21. Find the sum of the products, two at a time, of the coefficients in the expansion of $(1+x)^n$, when n is a positive integer.

22. If $(7+4\sqrt{3})^n=p+\beta$, where n and p are positive integers, and β a proper fraction, shew that $(1-\beta)(p+\beta)=1$.

23. If $c_0,\ c_1,\ c_2,\ \ldots\ldots\ c_n$ are the coefficients in the expansion of $(1+x)^n$, where n is a positive integer, shew that

$$c_1-\frac{c_2}{2}+\frac{c_3}{3}-\ldots\ldots+\frac{(-1)^{n-1}c_n}{n}=1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}.$$