6 Random Variable and Mathematical Expectation

Simeon-Denis Poisson

(June 21, 1781 – April 25, 1840)

Introduction

Between Sylvestre-François Lacroix in 1816 and Louis Bachelier in 1914, the concepts random variable and mean of a random variable were invented. Some authors contend that Simeon-Denis Poisson invented the concepts random variable and expected value. Simeon-Denis Poisson, a French mathematician known for his work on probability theory. Poisson's research was mainly on Probability. In 1838, Poisson published his ideas on probability theory, which also included what is now known as Poisson distribution. He published between 300 and 400 mathematical works including applications to electricity and magnetism, and astronomy.

Learning Objectives

After studying this chapter students are able to understand

- Why do we use random variable?
- Why do we need to define the random variable?
- the types of Random variables.
- the probability function.
- the distribution function.
- the nature of problems.
- the methods for studying random experiments with outcomes that can be described numerically.
- the real probabilistic computation.
- the concept of mathematical expectation for discrete and continuous random variables.
- the properties of mathematical expectation for discrete and continuous random variables.
- the determination of mean and variance for discrete and continuous random variables.
- and the practical applications of mathematical expectations for discrete and continuous random variables.

6.1. Random variable

Introduction:

Let the random experiment be the toss of a coin. When 'n' coins are tossed, one may be interested in knowing the number of heads obtained. When a pair of dice is thrown, one may seek information about the sum of sample points. Thus, we associate a real number with each outcome of an experiment. In other words, we are considering a function whose domain is the set of possible outcomes and whose range is subset of the set of real numbers. Such a function is called random variable.

In algebra, you learned about different variables like X or Y or any other letter in a particular problem. Thus in basic mathematics, a variable is an alphabetical character that represents an unknown number. A random variable is a variable that is subject to randomness, which means it could take on different values. In statistics, it is quite general to use X to denote a random variable and it takes on different values depending on the situation.

Some of the examples of random variable:

(i) Number of heads, if a coin is tossed 8 times.

(ii) The return on an investment in one-year period.

(iii) Faces on rolling a die.

(iv) Number of customers who arrive at a bank in the regular interval of one hour between 9.00 a.m and 4.30 p.m from Monday to Friday.

(v) The sale volume of a store on a particular day.

For instance, the random experiment 'E' consists of three tosses of a coin and the outcomes of this experiment forms the sample space is 'S'. Let X denotes the number of heads obtained.

Tossing a Coin
Fig.6.1

Here X is a real number connected with the outcome of a random experiment E. The details given below

Outcome (ω)	Values of $X=x$	Outcome (ω)	Values of $X=x$
(HHH)	3	(HTT)	1
(HHT)	2	(THT)	1
(HTH)	2	(TTH)	1
(THH)	2	(TTT)	0

i.e., $R_X = \{0, 1, 2, 3\}$

From the above said example, for each outcomes ω, there corresponds a real number $X(\omega)$. Since the points of the sample space 'S' corresponds to outcomes is defined for each $\omega \in S$.

6.1.1 Definition of a random variable

Definition 6.1

A random variable (r.v.) is a real valued function defined on a sample space S and taking values in $(-\infty, \infty)$ or whose possible values are numerical outcomes of a random experiment.

Note

(i) If x is a real number, the set of all ω in S such that $X(\omega)=x$ is, denoted by $X=x$. Thus $P(X=x) = P\{\omega : X(\omega) = x\}$.

(ii) $P(X \leq a) = P\{\omega : X(\omega) \in (-\infty, a]\}$ and $P(a < X \leq b) = P\{\omega : X(\omega) \in (a, b]\}$.

(iii) One-dimensional random variables will be denoted by capital letters, $X, Y, Z, \ldots$, etc. A typical outcome of the experiment will be denoted by ω. Thus $b \in X(\omega)$ represents the real number which the random variable X associates with the outcome τ. The values which $X, Y, Z, \ldots$, etc, can assume are denoted by lower case letters, *viz.*, $x, y, z, \ldots$, etc.

DO YOU KNOW?

(i) If X_1 and X_2 are random variables and C is a constant, then $CX_1, X_1 + X_2, X_1X_2, X_1 - X_2$ are also random variables.

(ii) If X is a random variable, then $(i)\frac{1}{X}$ and $(ii)|X|$ are also random variables.

Types of Random Variable:

Random variables are classified into two types namely discrete and continuous random variables. These are important for practical applications in the field of Mathematics and Statistics. The above types of random variable are defined with examples as follows.

6.1.2 Discrete random variable

Definition 6.2

A variable which can assume finite number of possible values or an infinite sequence of countable real numbers is called a discrete random variable.

Examples of discrete random variable:

- Marks obtained in a test.
- Number of red marbles in a jar.
- Number of telephone calls at a particular time.
- Number of cars sold by a car dealer in one month, etc.,

For instance, three responsible persons say, P_1, P_2 and P_3 are asked about their opinion in favour of building a model school in a certain district. Each person's response is recorded as Yes (Y) or No (N). Determine the random variable that could be of interest in this regard. The possibilities of the response are as follows

Table 6.1

Possibilities	P_1	P_2	P_3	Values of Random variable (Number of Yes those who are given)
1.	Y	Y	Y	3
2.	Y	Y	N	2
3.	Y	N	Y	2
4.	Y	N	N	1
5.	N	Y	Y	2
6.	N	Y	N	1
7.	N	N	Y	1
8.	N	N	N	0

Form the above table, the discrete random variable take values 0, 1, 2 and 3.

Probability Mass function

Definition 6.3

If X is a discrete random variable with distinct values $x_1, x_2, ..., x_n, ...,$ then the function, denoted by $P_X(x)$ and defined by

$$P_X(x) = p(x) = \begin{cases} P(X = x_i) = p_i = p(x_i) & if\ x = x_i, i = 1, 2, ..., n, ... \\ 0 & if\ x \neq x_i \end{cases}$$

This is defined to be the probability mass function or discrete probability function of X.

The probability mass function $p(x)$ must satisfy the following conditions

(i) $p(x_i) \geq 0\ \forall\, i,$ (ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

Example 6.1

The number of cars in a household is given below.

No. of cars	0	1	2	3	4
No. of Household	30	320	380	190	80

Estimate the probability mass function. Verify $p(x_i)$ is a probability mass function.

Solution:

Let X be the number of cars

Table 6.2

$X = x_i$	Number of Household	$p(x_i)$
0	30	0.03
1	320	0.32
2	380	0.38
3	190	0.19
4	80	0.08
Total	1000	1.00

(i) $p(x_i) \geq 0 \ \forall \ i$ and

(ii) $\sum_{i=1}^{\infty} p(x_i)$

$= p(0) + p(1) + p(2) + p(3) + p(4)$

$= 0.03+0.32+0.38+0.19+0.08 = 1$

Hence $p(x_i)$ is a probability mass function.

Note

For $X = 0$, the probability 0.03, comes from 30/1000, the other probabilities are estimated similarly.

Example 6.2

A random variable X has the following probability function

Values of X	0	1	2	3	4	5	6	7
$p(x)$	0	a	$2a$	$2a$	$3a$	a^2	$2a^2$	$7a^2 + a$

(i) Find a, Evaluate (ii) $P(X < 3)$, (iii) $P(X > 2)$ and (iv) $P(2 < X \leq 5)$.

Solution:

(i) Since the condition of probability mass function is

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

$$\therefore \sum_{i=0}^{7} p(x_i) = 1$$

$0+a+2a+2a+3a+a^2+2a^2+7a^2+a = 1$

$10a^2+9a-1 = 0$

$(10a-1)(a+1) = 0$

$a = \frac{1}{10}$ and -1

Since $p(x)$ cannot be negative, $a = -1$ is not applicable. Hence, $a = \frac{1}{10}$

(ii) $P(X<3) = P(X=0)+P(X=1)+P(X=2)$

$= 0+a+2a$

$= 3a$

$= \frac{3}{10} \quad \left(\because a = \frac{1}{10}\right)$

(iii) $P(X>2) = 1 - P(X \leq 2)$

$= 1 - \left[P(X=0)+P(X=1)+P(X=2)\right]$

$= 1 - \frac{3}{10}$

$= \frac{7}{10}$

(iv) $P(2 < X \leq 5) = P(X=3)+P(X=4)+P(X=5)$

$= 2a + 3a + a^2$

$= 5a + a^2$

$= \frac{5}{10} + \frac{1}{100} = \frac{51}{100}$

Example 6.3

If $p(x) = \begin{cases} \frac{x}{20}, & x = 0,1,2,3,4,5 \\ 0, & otherwise \end{cases}$

Find (i) $P(X<3)$ and (ii) $P(2 < X \leq 4)$

Solution:

$P(X<3) = P(X=0)+P(X=1)+P(X=2)$

$= 0 + \frac{1}{20} + \frac{2}{20} = \frac{3}{20}$

$P(2 < X \leq 4) = P(X=3)+P(X=4)$

$= \frac{3}{20} + \frac{4}{20} = \frac{7}{20}$

Example 6.4

If you toss a fair coin three times, the outcome of an experiment consider as random variable which counts the number of heads on the upturned faces. Find out the probability mass function and check the properties of the probability mass function.

Solution:

Let X is the random variable which counts the number of heads on the upturned faces. The outcomes are stated below

Outcomes	Values of X	Outcomes	Values of X
(HHH)	3	(HTT)	1
(HHT)	2	(THT)	1
(HTH)	2	(TTH)	1
(THH)	2	(TTT)	0

These values are summarized in the following probability table.

Table 6.3

Value of X	0	1	2	3	Total
$p(x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\sum_{i=0}^{3} p(x_i)=1$

(i) $p(x_i) \geq 0 \; \forall \; i$ and

(ii) $\sum_{i=0}^{3} p(x_i)=1$

Hence, $p(x_i)$ is a probability mass function.

Example 6.5

Two unbiased dice are thrown simultaneously and sum of the upturned faces considered as random variable. Construct a probability mass function.

Fig.6.2

Solution:

$$\text{Sample space } (S)=\begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

Total outcomes : $n(S) = 36$

Table 6.4

Out comes	Sum of the upturned faces (X)	$P_X(x)$
(1,1)	2	$\frac{1}{36}$
(1,2) (2,1)	3	$\frac{2}{36}$
(1,3) (2,2) (3,1)	4	$\frac{3}{36}$
(1,4) (2,3) (3,2) (4,1)	5	$\frac{4}{36}$
(1,5) (2,4) (3,3) (4,2) (5,1)	6	$\frac{5}{36}$
(1,6) (2,5) (3,4) (4,3) (5,2) (6,1)	7	$\frac{6}{36}$
(2,6) (3,5) (4,4) (5,3) (6,2)	8	$\frac{5}{36}$
(3,6) (4,5) (5,4) (6,3)	9	$\frac{4}{36}$
(4,6) (5,5) (6,4)	10	$\frac{3}{36}$
(5,6) (6,5)	11	$\frac{2}{36}$
(6,6)	12	$\frac{1}{36}$

Discrete distribution function

Definition 6.4

The discrete cumulative distribution function or distribution function of a real valued discrete random variable X takes the countable number of points $x_1, x_2, ...$ with corresponding probabilities $p(x_1), p(x_2), ...$ and then the cumulative distribution function is defined by

$$F_X(x)=P(X \leq x), \textit{ for all } x \in R$$

$$i.e., F_X(x)=\sum_{x_i \leq x} p(x_i)$$

For instance, suppose we have a family of two children. The sample space

S = {bb, bg, gb, gg}, where b =boy and g = girl

Let X be the random variable which counts the number of boys. Then, the values (X) corresponding to the sample space are 2, 1, 1, and 0.

Hence, the probability mass function of X is

$X=x$	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Then, we can form a cumulative distribution function of X is

$X=x$	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$F_X(x)=P(X\leq x)$	$\frac{1}{4}$	$\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$	$\frac{3}{4}+\frac{1}{4}=1$

Example 6.6

A coin is tossed thrice. Let X be the number of observed heads. Find the cumulative distribution function of X .

Solution:

The sample space (S) ={ (HHH), (HHT), (HTH), (HTT),(THH),(THT),(TTH),(TTT)}

X takes the values: 3, 2, 2, 1, 2, 1, 1, and 0

Range of $X(R_x)$	0	1	2	3
$P_x(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$F_x(x)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	1

Thus, we have
$$F_X(x)=\begin{cases} 0, & for\ x<0 \\ \frac{1}{8}, & for\ 0\leq x<1 \\ \frac{4}{8}, & for\ 1\leq x<2 \\ \frac{7}{8}, & for\ 2\leq x<3 \\ 1, & for\ x\leq 3 \end{cases}$$

Example 6.7

Construct the distribution function for the discrete random variable X whose probability distribution is given below. Also draw a graph of p(x) and F(x).

$X=x$	1	2	3	4	5	6	7
$P(x)$	0.10	0.12	0.20	0.30	0.15	0.08	0.05

Solution

From the values of $p(x)$ given in the probability distribution, we obtain

$$F(1)=P(x\leq 1)=P(1)=0.10$$

$$F(2)=P(x\leq 2)=P(1)+P(2)$$
$$=0.10+0.12=0.22$$

$$F(3)=P(x\leq 3)=P(1)+P(2)+P(3)$$
$$=F(2)+P(3)$$
$$=0.22+0.20$$
$$=0.42$$

$$F(4)=F(3)+P(4)$$
$$=0.42+0.30$$
$$=0.72$$

$$F(5)=F(4)+P(5)$$
$$=0.72+0.15$$
$$=0.87$$

$$F(6)=F(5)+P(6)$$
$$=0.87+0.08$$
$$=0.95$$

$$F(7)=F(6)+P(7)$$
$$=0.95+0.05$$
$$=1.00$$

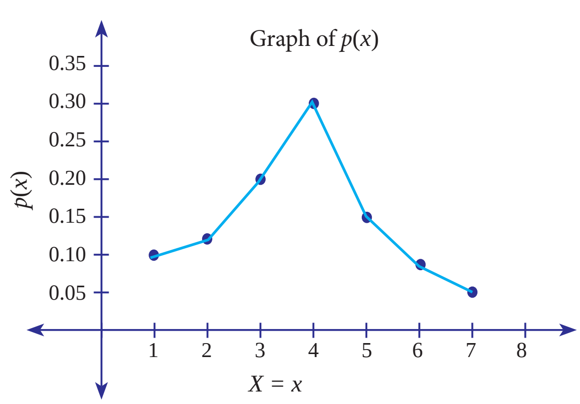

Fig.6.3

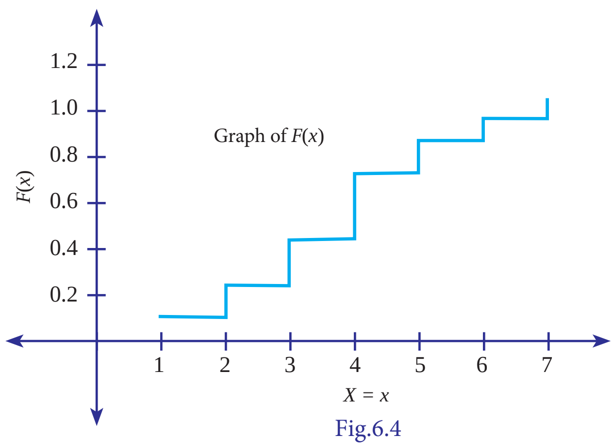

Fig.6.4

$$F(x) \text{ is } F_X(x)=\begin{cases} 0, & if\ x<1 \\ 0.10, & if\ x\le 1 \\ 0.22, & if\ x\le 2 \\ 0.42, & if\ x\le 3 \\ 0.72, & if\ x\le 4 \\ 0.87, & if\ x\le 5 \\ 0.95, & if\ x\le 6 \\ 1, & if\ x\le 7 \end{cases}$$

6.1.3 Continuous random variable

Definition 6.5

A random variable X which can take on any value (integral as well as fraction) in the interval is called continuous random variable.

Examples of continuous random variable

- The amount of water in a 10 ounce bottle.
- The speed of a car.
- Electricity consumption in kilowatt hours.
- Height of people in a population.
- Weight of students in a class.
- The length of time taken by a truck driver to go from Chennai to Madurai, etc.

Probability density function

Definition 6.6

The probability that a random variable X takes a value in the interval $[t_1, t_2]$ (open or closed) is given by the integral of a function called the probability density function $f_X(x)$:

$$P(t_1 \le X \le t_2) = \int_{t_1}^{t_2} f_X(x)dx.$$

Other names that are used instead of probability density function include density function, continuous probability function, integrating density function.

The probability density functions $f_X(x)$ or simply by $f(x)$ must satisfy the following conditions.

(i) $f(x) \ge 0\ \forall\ x$ and (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$.

Example 6.8

A continuous random variable X has the following p.d.f $f(x) = ax$, $0 \le x \le 1$

Determine the constant a and also find $P\left[X \le \frac{1}{2}\right]$

Solution:

We know that

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^1 ax\ dx = 1 \Rightarrow a\int_0^1 xdx = 1$$

$$\Rightarrow a\left(\frac{x^2}{2}\right)_0^1 = 1$$

$$\Rightarrow \frac{a}{2}(1-0) = 1$$

$$\Rightarrow a = 2$$

$$P\left[x \le \frac{1}{2}\right] = \int_{-\infty}^{\frac{1}{2}} f(x)dx$$

$$= \int_0^{\frac{1}{2}} axdx \quad = \int_0^{\frac{1}{2}} 2xdx \quad = \frac{1}{4}$$

Example 6.9

A continuous random variable X has p.d.f $f(x) = 5x^4, 0 \le x \le 1$ Find a_1 and a_2 such that (i) $P[X \le a_1] = P[X > a_1]$ (ii) $P[X > a_2] = 0.05$

Solution

(i) Since $P[X \le a_1] = P[X > a_1]$

$$P[X \le a_1] = \frac{1}{2}$$

i.e., $\int_0^{a_1} f(x)dx = \frac{1}{2}$

i.e., $\int_0^{a_1} 5x^4 dx = \frac{1}{2}$

$$5\left[\frac{x^5}{5}\right]_0^{a_1} = \frac{1}{2}$$

$$a_1 = (0.5)^{\frac{1}{5}}$$

(ii) $P[X > a_2] = 0.05$

$$\int_{a_2}^{1} f(x)dx = 0.05$$

$$\int_{a_2}^{1} 5x^4 dx = 0.05$$

$$5\left[\frac{x^5}{5}\right]_{a_2}^{1} = 0.05$$

$$a_2 = [0.95]^{\frac{1}{5}}$$

Continuous distribution function

Definition 6.7

If X is a continuous random variable with the probability density function $f_X(x)$, then the function $F_X(x)$ is defined by

$$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f(t)dt, -\infty < x < \infty$$

is called the distribution function (d.f) or sometimes the cumulative distribution function (c.d.f) of the continuous random variable X.

Properties of cumulative distribution function

The function $F_X(x)$ or simply $F(x)$ has the following properties

(i) $0 \le F(x) \le 1, -\infty < x < \infty$

(ii) $F(-\infty) = \lim_{x\to -\infty} F(x) = 0$ and $F(+\infty) = \lim_{x\to\infty} F(x) = 1$.

(iii) $F(\cdot)$ is a monotone, non-decreasing function; that is, $F(a) \le F(b)$ for $a < b$.

(iv) $F(\cdot)$ is continuous from the right; that is, $\lim_{h\to 0} F(x+h) = F(x)$.

(v) $F'(x) = \frac{d}{dx}F(x) = f(x) \ge 0$

(vi) $F'(x) = \frac{d}{dx}F(x) = f(x) \Rightarrow dF(x) = f(x)dx$, $dF(x)$ is known as probability differential of X.

(vii) $P(a \le x \le b) = \int_a^b f(x)dx = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$

$$= P(X \le b) - P(X \le a)$$

$$= F(b) - F(a)$$

Example 6.10

Suppose, the life in hours of a radio tube has the following p.d.f

$$f(x) = \begin{cases} \frac{100}{x^2}, when\, x \ge 100 \\ 0, \quad when\, x < 100 \end{cases}$$

Find the distribution function.

Solution:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

$$= \int_{100}^{x} \frac{100}{t^2} dt, x \ge 100$$

$$= \left[\frac{100}{-t}\right]_{100}^{x}, x \ge 100$$

$$F(x) = \left[1 - \frac{100}{x}\right], x \ge 100$$

Example 6.11

The amount of bread (in hundreds of pounds) x that a certain bakery is able to sell in a day is found to be a numerical valued random phenomenon, with a probability function specified by the probability density function $f(x)$ is given by

$$f(x) = \begin{cases} Ax, for\, 0 \le x < 10 \\ A(20-x), for\, 10 \le x < 20 \\ 0, \quad otherwise \end{cases}$$

(a) Find the value of A.

(b) What is the probability that the number of pounds of bread that will be sold tomorrow is

(i) More than 10 pounds,

(ii) Less than 10 pounds, and

(iii) Between 5 and 15 pounds?

Solution:

(a) We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^{10} Axdx + \int_{10}^{20} A(20-x)dx = 1$$

$$A\left\{\left[\frac{x^2}{2}\right]_0^{10} + \left[20x - \frac{x^2}{2}\right]_{10}^{20}\right\} = 1$$

$$A\left[(50-0)+(400-200)-(200-50)\right]=1$$

$$A = \frac{1}{100}$$

(b) (i) The probability that the number of pounds of bread that will be sold tomorrow is more than 10 pounds is given by

$$P(10 \le X \le 20) = \int_{10}^{20} \frac{1}{100}(20-x)\,dx$$

$$= \frac{1}{100}\left[20x - \frac{x^2}{2}\right]_{10}^{20}$$

$$= \frac{1}{100}\left[(400-200)-(200-50)\right]$$

$$= 0.5$$

(ii) The probability that the number of pounds of bread that will be sold tomorrow is less than 10 pounds, is given by

$$P(0 \le X < 10) = \int_0^{10} \frac{1}{100} xdx$$

$$= \frac{1}{100}\left[\frac{x^2}{2}\right]_0^{10}$$

$$= \frac{1}{100}(50-0)$$

$$= 0.5$$

(iii) The probability that the number of pounds of bread that will be sold tomorrow is between 5 and 15 pounds is

$$P(5 \le X \le 15) = \int_5^{10} \frac{1}{100} xdx + \int_{10}^{15} \frac{1}{100}(20-x)dx$$

$$= \frac{1}{100}\left[\frac{x^2}{2}\right]_5^{10} + \frac{1}{100}\left[20x - \frac{x^2}{2}\right]_{10}^{15}$$

$$= 0.75$$

Exercise 6.1

1. Construct cumulative distribution function for the given probability distribution.

X	0	1	2	3
$P(X=x)$	0.3	0.2	0.4	0.1

2. Let X be a discrete random variable with the following p.m.f

$$p(x) = \begin{cases} 0.3 & for\ x = 3 \\ 0.2 & for\ x = 5 \\ 0.3 & for\ x = 8 \\ 0.2 & for\ x = 10 \\ 0 & otherwise \end{cases}$$

Find and plot the c.d.f. of X.

3. The discrete random variable X has the following probability function

$$P(X=x) = \begin{cases} kx & x = 2,4,6 \\ k(x-2) & x = 8 \\ 0 & otherwise \end{cases}$$

where k is a constant. Show that $k = \frac{1}{18}$

4. The discrete random variable X has the probability function

X	1	2	3	4
$P(X=x)$	k	$2k$	$3k$	$4k$

Show that $k = 0{\cdot}1$.

5. Two coins are tossed simultaneously. Getting a head is termed as success. Find the probability distribution of the number of successes.

6. A continuous random variable X has the following probability function

Value of $X=x$	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Find k

(ii) Evaluate $p(x<6)$, $p(x \geq 6)$ and $p(0<x<5)$

(iii) If $P(X \leq x) > \frac{1}{2}$, then find the minimum value of x.

7. The distribution of a continuous random variable X in range (–3, 3) is given by p.d.f.

$$f(x)=\begin{cases} \frac{1}{16}(3+x)^2, & -3\leq x\leq -1 \\ \frac{1}{16}(6-2x^2), & -1\leq x\leq 1 \\ \frac{1}{16}(3-x)^2, & 1\leq x\leq 3 \end{cases}$$

Verify that the area under the curve is unity.

8. A continuous random variable X has the following distribution function

$$F(x)=\begin{cases} 0 & ,if\ x\leq 1 \\ k(x-1)^4 & ,if\ 1<x\leq 3 \\ 1 & ,if\ x>3 \end{cases}$$

Find (i) k and (ii) the probability density function.

9. The length of time (in minutes) that a certain person speaks on the telephone is found to be random phenomenon, with a probability function specified by the probability density function $f(x)$ as

$$f(x)=\begin{cases} Ae^{-x/5} & , for\ x\geq 0 \\ 0 & ,otherwise \end{cases}$$

(a) Find the value of A that makes $f(x)$ a p.d.f.

(b) What is the probability that the number of minutes that person will talk over the phone is (i) more than 10 minutes, (ii) less than 5 minutes and (iii) between 5 and 10 minutes.

10. Suppose that the time in minutes that a person has to wait at a certain station for a train is found to be a random phenomenon with a probability function specified by the distribution function

$$F(x)=\begin{cases} 0, & for\ x\leq 0 \\ \frac{x}{2}, & for\ 0\leq x<1 \\ \frac{1}{2}, & for\ 1\leq x<2 \\ \frac{x}{4}, & for\ 2\leq x<4 \\ 1, & for\ x\geq 4 \end{cases}$$

(a) Is the distribution function continuous? If so, give its probability density function?

(b) What is the probability that a person will have to wait (i) more than 3 minutes, (ii) less than 3 minutes and (iii) between 1 and 3 minutes?

11. Define random variable.

12. Explain what are the types of random variable?

13. Define discrete random variable.

14. What do you understand by continuous random variable?

15. Describe what is meant by a random variable.

16. Distinguish between discrete and continuous random variable.

17. Explain the distribution function of a random variable.

18. Explain the terms (i) probability mass function, (ii) probability density function and (iii) probability distribution function.

19. What are the properties of (i) discrete random variable and (ii) continuous random variable?

20. State the properties of distribution function.

6.2. Mathematical Expectation

Introduction

An extremely useful concept in problems involving random variables or distributions is that of expectation. Random variables can be characterized and dealt with effectively for practical purposes by consideration of quantities called their expectation. The concept of mathematical expectation arose in connection with games of chance. For example, a gambler might be interested in his average winnings at a game, a businessman in his average profits on a product, and so on. The average value of a random phenomenon is also termed as its Mathematical expectation or expected value. In the following sections, we will define and study the concept of mathematical expectation for both discrete and continuous random variables, which will be used in the following subsection.

6.2.1 Expected value and Variance

Expected value

The expected value is a weighted average of the values of a random variable may assume. The weights are the probabilities.

Definition 6.8

Let X be a discrete random variable with probability mass function (p.m.f.) $p(x)$. Then, its expected value is defined by

$$E(X) = \sum_x x\, p(x) \qquad \ldots (1)$$

If X is a continuous random variable and $f(x)$ is the value of its probability density function at x, the expected value of X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx \qquad \ldots (2)$$

Note

- In (1), $E(X)$ is defined to be the indicated series provided that the series is absolutely convergent; otherwise, we say that the mean does not exist.
- In (1), $E(X)$ is an "average" of the values that the random variable takes on, where each value is weighted by the probability that the random variable is equal to that value. Values that are more probable receive more weight.
- In (2), $E(X)$ is defined to be the indicated integral if the integral exists; otherwise, we say that the mean does not exist.
- In (2), $E(X)$ is an "average" of the values that the random variable takes on, where each value x is multiplied by the approximate probability that X equals the value x, namely $f_X(x)dx$ and then integrated over all values.

Do You Know?

- $E(X)$ is the center of gravity or centroid of the unit mass that is determined by the density function of X. So the mean of X is measure of where the values of the random variable X are "centered".
- The mean of X, denoted by φ_x or $E(X)$.

Variance

The variance is a weighted average of the squared deviations of a random variable from its mean. The weights are the probabilities. The mean of a random variable X, defined in (1) and (2), was a measure of central location of the density of X. The variance of a random variable X will be a measure of the spread or dispersion of the density of X or simply the variability in the values of a random variable.

Definition 6.9

The variance of X is defined by

$$Var(X)=\sum\left[x-E(X)\right]^2 p(x) \quad ...(3)$$

if X is discrete random variable with probability mass function $p(x)$.

$$Var(X)=\int_{-\infty}^{\infty}\left[x-E(X)\right]^2 f_X(x)\,dx \quad ...(4)$$

if X is continuous random variable with probability density function $f_X(x)$.

Definition 6.10

Expected value of $[X-E(X)]^2$ is called the variance of the random variable.

i.e., $Var(X)=E\left[X-E(X)\right]^2=E\left(X^2\right)-\left[E(X)\right]^2$ ... (5)

where

$$E\left(X^2\right)=\begin{cases}\sum_x x^2\,p(x), & \text{if } X \text{ is Discrete Random Variable}\\ \int_{-\infty}^{\infty} x^2 f(x)\,dx, & \text{if } X \text{ is Continuous Random Variable}\end{cases}$$

Note

- In the following examples, variance will be found using definition 6.10.
- The variances are defined only if the series in (3) is convergent or if the integrals in (4) exist.
- If X is a random variable, the standard deviation of X (S.D(X)), denoted by σ_X, is defined as $+\sqrt{Var[X]}$.
- The variance of X, denoted by $\sigma_X{}^2$ or $Var(X)$ or $V(X)$

DO YOU KNOW?

Mean is the center of gravity of a density; similarly, variance represents the moment of inertia of the same density with respect to a perpendicular axis through the center of gravity.

6.2.2 Properties of Mathematical expectation

(i) $E(a)=a$, where 'a' is a constant

(ii) $E(aX)=aE(X)$

(iii) $E(aX+b)=aE(X)+b$, where 'a' and 'b' are constants.

(iv) If $X\geq 0$, then $E(X)\geq 0$

(v) $V(a)=0$

(vi) If X is random variable, then

$$V(aX+b)=a^2V(X)$$

Concept of moments

The moments (or raw moments) of a random variable or of a distribution are the expectations of the powers of the random variable which has the given distribution.

Definition 6.11

If X is a random variable, then the r^{th} moment of X, usually denoted by φ'_r, is defined as φ'_r =

$$E\left(X^r\right)=\begin{cases}\sum_x x^r\,p(x), & \text{for discrete random variable}\\ \int_{-\infty}^{\infty} x^r f(x)\,dx, & \text{for continuous random variable}\end{cases}$$

provided the expectation exists.

Definition 6.12

If X is a random variable, the r^{th} central moment of X about a is defined as $E\left[(X-a)^r\right]$. If $a=\varphi_x$, we have the r^{th} central moment of X about φ_x, denoted by φ_r, which is

$$\varphi_r = E[(X-\varphi_X)^r]$$

Note

- $\varphi'_1 = E(X) = \varphi_X$, the mean of X.
- $\varphi_1 = E[X-\varphi_X] = 0$.
- $\varphi_2 = E[(X-\varphi_X)^2]$, the variance of X.
- All odd moments of X about φ_X are 0 if the density function of X is symmetrical about φ_X, provided such moments exist.

Example 6.12

Determine the mean and variance of the random variable X having the following probability distribution.

$X = x$	1	2	3	4	5	6	7	8	9	10
$P(x)$	0.15	0.10	0.10	0.01	0.08	0.01	0.05	0.02	0.28	0.20

Solution:

Mean of the random variable

$$X = E(X) = \sum_x x P_X(x)$$

$$= (1 \times 0.15) + (2 \times 0.10) + (3 \times 0.10) + (4 \times 0.01) + (5 \times 0.08) + (6 \times 0.01) + (7 \times 0.05) + (8 \times 0.02) + (9 \times 0.28) + (10 \times 0.20)$$

$$E(X) = 6.18$$

$$E(X^2) = \sum_x x^2 P_X(x)$$

$$= (1^2 \times 0.15) + (2^2 \times 0.10) + (3^2 \times 0.10) + (4^2 \times 0.01) + (5^2 \times 0.08) + (6^2 \times 0.01) + (7^2 \times 0.05) + (8^2 \times 0.02) + (9^2 \times 0.28) + (10^2 \times 0.20).$$

$$= 50.38$$

Variance of the Random Variable

$$X = V(X) = E(X^2) - [E(X)]^2$$

$$= 50.38 - (6.56)^2$$

$$= 12.19$$

Therefore, the mean and variance of the given discrete distribution are 6.18 and 12.19 respectively.

Example 6.13

Six men and five women apply for an executive position in a small company. Two of the applicants are selected for an interview. Let X denote the number of women in the interview pool. We have found the probability mass function of X.

$X = x$	0	1	2
$P(x)$	$\frac{2}{11}$	$\frac{5}{11}$	$\frac{4}{11}$

How many women do you expect in the interview pool?

Solution:

Expected number of women in the interview pool is

$$E(X) = \sum_x x P_X(x)$$

$$= \left[\left(0 \times \frac{2}{11}\right) + \left(1 \times \frac{5}{11}\right) + \left(2 \times \frac{4}{11}\right)\right]$$

$$= \frac{13}{11}$$

Example 6.14

Determine the mean and variance of a discrete random variable, given its distribution as follows.

$X = x$	1	2	3	4	5	6
$F_x(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

Solution

From the given data, you first calculate the probability distribution of the random variable. Then using it you calculate mean and variance.

X	$p(x)$
1	$F(1) = \frac{1}{6}$
2	$F(2) - F(1) = \frac{2}{6} - \frac{1}{6} = \frac{1}{6}$
3	$F(3) - F(2) = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$
4	$F(4) - F(3) = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$
5	$F(5) - F(4) = \frac{5}{6} - \frac{4}{6} = \frac{1}{6}$
6	$F(6) - F(5) = 1 - \frac{5}{6} = \frac{1}{6}$

The probability mass function is

$X = x$	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Mean of the random variable X

$$= E(X) = \sum_x x\, P_X(x)$$

$$= \left(1\times\frac{1}{6}\right)+\left(2\times\frac{1}{6}\right)+\left(3\times\frac{1}{6}\right)+\left(4\times\frac{1}{6}\right)+\left(5\times\frac{1}{6}\right)+\left(6\times\frac{1}{6}\right)$$

$$= \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{7}{2}$$

$$E\left(X^2\right) = \sum_x x^2 P_X(x)$$

$$= \left(1^2\times\frac{1}{6}\right)+\left(2^2\times\frac{1}{6}\right)+\left(3^2\times\frac{1}{6}\right)+\left(4^2\times\frac{1}{6}\right)+\left(5^2\times\frac{1}{6}\right)+\left(6^2\times\frac{1}{6}\right)$$

$$= \frac{1}{6}\left(1^2+2^2+3^2+4^2+5^2+6^2\right)$$

$$= \frac{91}{6}$$

Variance of the Random Variable

$$X = V(X) = E\left(X^2\right)-[E(X)]^2$$

$$= \frac{91}{6}-\left(\frac{7}{2}\right)^2$$

$$= \frac{35}{12}$$

Example 6.15

The following information is the probability distribution of successes.

No. of Successes	0	1	2
Probability	$\frac{6}{11}$	$\frac{9}{22}$	$\frac{1}{22}$

Determine the expected number of success.

Solution:

Expected number of success is

$$E(X) = \sum_x x\, P_X(x)$$

$$= \left(0\times\frac{6}{11}\right)+\left(1\times\frac{9}{22}\right)+\left(2\times\frac{1}{22}\right)$$

$$= \frac{11}{22} = 0.5$$

Therefore, the expected number of success is 0.5. Approximately one success.

Example 6.16

An urn contains four balls of red, black, green and blue colours. There is an equal probability of getting any coloured ball. What is the expected value of getting a blue ball out of 30 experiments with replacement?

Solution:

Probability of getting a blue ball (p) = $\frac{1}{4}$

= 0.25

Total experiments (N) = 30

Expected value = Number of experiments × Probability

= $N \times p$

= 30 × 0.25

= 7.50

Therefore, the expected value of getting blue ball is approximately 8.

Example 6.17

A fair die is thrown. Find out the expected value of its outcomes.

Solution:

If the random variable X is the top face of a tossed, fair, six sided die, then the probability mass function of X is

$$P_X(x) = \frac{1}{6}, \text{ for } x = 1,2,3,4,5 \text{ and } 6$$

The average toss, that is, the expected value of X is

$$E(X) = \sum_x x\, P_X(x)$$

$$= \left(1\times\frac{1}{6}\right)+\left(2\times\frac{1}{6}\right)+\left(3\times\frac{1}{6}\right)+\left(4\times\frac{1}{6}\right)$$

$$+\left(5\times\frac{1}{6}\right)+\left(6\times\frac{1}{6}\right)$$

$$= \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2} = 3.5$$

Therefore, the expected toss of a fair six sided die is 3.5.

Example 6.18

Suppose the probability mass function of the discrete random variable is

$X = x$	0	1	2	3
$p(x)$	0.2	0.1	0.4	0.3

What is the value of $E(3X + 2X^2)$?

Solution:

$$E(X) = \sum_x x P_X(x)$$
$$= (0 \times 0.2) + (1 \times 0.1) + (2 \times 0.4) + (3 \times 0.3)$$
$$= 1.8$$
$$E(X^2) = \sum_x x^2 P_X(x)$$
$$= (0^2 \times 0.2) + (1^2 \times 0.1) + (2^2 \times 0.4) + (3^2 \times 0.3)$$
$$= 4.4$$
$$E(3X + 2X^2) = 3E(X) + 2E(X^2)$$
$$= (3 \times 1.8) + (2 \times 4.4)$$
$$= 14.2$$

Example 6.19

Consider a random variable X with probability density function

$$f(x) = \begin{cases} 4x^3, if\ 0 < x < 1 \\ 0,\ otherwise \end{cases}$$

Find $E(X)$ and $V(X)$.

Solution:

We know that,

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$
$$= \int_0^1 x\, 4x^3\, dx$$
$$= 4\left[\frac{x^5}{5}\right]_0^1$$
$$E(X) = \frac{4}{5}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$
$$= \int_0^1 x^2\, 4x^3\, dx$$
$$= 4\left[\frac{x^6}{6}\right]_0^1$$
$$= \frac{4}{6} = \frac{2}{3}$$
$$V(X) = E(X^2) - [E(X)]^2$$
$$= \frac{4}{6} - \left[\frac{4}{5}\right]^2$$
$$= \frac{2}{75}$$

Example 6.20

If $f(x)$ is defined by $f(x) = ke^{-2x}, 0 \le x < \infty$ is a density function. Determine the constant k and also find mean.

Solution:

We know that

$\int_{-\infty}^{\infty} f(x) dx = 1$, since $f(x)$ is a density function.

$$\int_0^{\infty} k e^{-2x} dx = 1$$
$$k \int_0^{\infty} e^{-2x} dx = 1$$
$$k\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = 1$$
$$\Rightarrow k = 2$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$
$$= \int_0^{\infty} xk e^{-2x} dx = 2\int_0^{\infty} xe^{-2x} dx$$
$$= 2\left\{\left[\frac{xe^{-2x}}{-2}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-2x}}{-2} dx\right\}$$
$$\left(\because \int udv = uv - \int vdu\right)$$
$$= \int_0^{\infty} e^{-2x} dx = \frac{1}{2}$$

Example 6.21

The time to failure in thousands of hours of an important piece of electronic equipment used in a manufactured DVD player has the density function.

$$f(x)=\begin{cases}3e^{-3x}, & x>0\\ 0, & otherwise\end{cases}$$

Find the expected life of the piece of equipment.

Solution:

We know that,

$$E(X)=\int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$=\int_{0}^{\infty} x\,3e^{-3x}dx$$

$$=3\int_{0}^{\infty} x\,e^{-3x}dx$$

$$=3\left\{\left[x\frac{e^{-3x}}{-3}\right]_0^{\infty}-\int_0^{\infty}\left(\frac{e^{-3x}}{-3}\right)dx\right\}\quad\left(\because \int u\,dv=uv-\int v\,du\right)$$

$$=\int_{0}^{\infty} e^{-3x}dx$$

$$=\frac{1}{3}$$

Therefore, the expected life of the piece of equipment is $\frac{1}{3}$ hrs (in thousands).

Example 6.22

A commuter train arrives punctually at a station every 25 minutes. Each morning, a commuter leaves his house and casually walks to the train station. Let X denote the amount of time, in minutes, that commuter waits for the train from the time he reaches the train station. It is known that the probability density function of X is

$$f(x)=\begin{cases}\frac{1}{25}, & for\, 0<x<25\\ 0, & otherwise.\end{cases}$$

Obtain and interpret the expected value of the random variable X.

Solution:

Expected value of the random variable is

$$E(X)=\int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$=\int_{0}^{25} x\frac{1}{25}dx$$

$$=\frac{1}{25}\int_{0}^{25} x\,dx$$

$$=\frac{1}{25}\left[\frac{x^2}{2}\right]_0^{25}$$

$$=12.5$$

Therefore, the expected waiting time of the commuter is 12.5 minutes.

Example 6.23

Suppose the life in hours of a radio tube has the probability density function

$$f(x)=\begin{cases}e^{-\frac{x}{100}}, & when\, x\geq 100\\ 0, & when\, x<100\end{cases}$$

Find the mean of the life of a radio tube.

Solution:

We know that, the expected random variable

$$E(X)=\int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$=\int_{100}^{\infty} x\,e^{-\frac{x}{100}}dx$$

$$=\left\{\left[x\left(\frac{e^{-\frac{x}{100}}}{-\frac{1}{100}}\right)\right]_{100}^{\infty}-\int_{100}^{\infty}\left(\frac{e^{-\frac{x}{100}}}{-\frac{1}{100}}\right)dx\right\}$$

$$\left(\because \int u\,dv=uv-\int v\,du\right)$$

$$=\left[(10000)\left(e^{-1}\right)+(10000)\left(e^{-1}\right)\right]$$

$$=\left[(10000)(0.3679)+(10000)(0.3679)\right]$$

$$=7358\ hours$$

Therefore, the mean life of a radio tube is 7,358 hours.

Example 6.24

The probability density function of a random variable X is

$$f(x)=ke^{-|x|}, -\infty<x<\infty$$

Find the value of k and also find mean and variance for the random variable.

Solution

We know that,

$$\int_{-\infty}^{\infty} f(x)dx=1$$

$$\int_{-\infty}^{\infty} ke^{-|x|}dx=1$$

$$k\int_{-\infty}^{\infty} e^{-|x|}dx=1$$

$$2k\int_{0}^{\infty} e^{-x}dx=1$$

($\because$ $e^{-|x|}$ is an even function)

$$2k\left[\frac{e^{-x}}{-1}\right]_0^{\infty}=1$$

$$k=\frac{1}{2}$$

Mean of the random variable is

$$E(X)=\int_{-\infty}^{\infty} x\,f(x)dx$$

$$E(X)=\int_{-\infty}^{\infty} x\,ke^{-|x|}dx$$

($\because$ $xe^{-|x|}$ is an odd function of x)

$$=\frac{1}{2}\int_{-\infty}^{\infty} xe^{-|x|}dx$$

$$=0$$

$$E\left(X^2\right)=\int_{-\infty}^{\infty} x^2 f(x)dx$$

$$=\int_{-\infty}^{\infty} x^2ke^{-|x|}dx$$

$$=\frac{1}{2}\int_{-\infty}^{\infty} x^2e^{-|x|}dx$$

$$=\int_{0}^{\infty} x^2e^{-x}dx \quad (\because\ e^{-|x|} \text{ is an even function})$$

$$=\Gamma(3)\left(\because \Gamma(\alpha)=\int_0^{\infty} x^{\alpha-1}e^{-x}dx, \alpha>0; \Gamma(n)=(n-1)!\right)$$

$$=2$$

$$V(X)=E\left(X^2\right)-\left[E(X)\right]^2$$

$$=2-[0]^2$$

$$=2$$

Exercise 6.2

1. Find the expected value for the random variable of an unbiased die

2. Let X be a random variable defining number of students getting A grade. Find the expected value of X from the given table

$X=x$	0	1	2	3
$P(X=x)$	0.2	0.1	0.4	0.3

3. The following table is describing about the probability mass function of the random variable X

x	3	4	5
$P(x)$	0.2	0.3	0.5

Find the standard deviation of x.

4. Let X be a continuous random variable with probability density function

$$f_X(x)=\begin{cases}2x, & 0\le x\le 1\\ 0, & otherwise\end{cases}$$

Find the expected value of X.

5. Let X be a continuous random variable with probability density function

$$f(x)=\begin{cases}\dfrac{3}{x^4}, & x\ge 1\\ 0, & otherwise\end{cases}$$

Find the mean and variance of X.

6. In an investment, a man can make a profit of ₹ 5,000 with a probability of 0.62 or

a loss of ₹ 8,000 with a probability of $0\cdot38$. Find the expected gain.

7. What are the properties of Mathematical expectation?

8. What do you understand by Mathematical expectation?

9. How do you define variance in terms of Mathematical expectation?

10. Define Mathematical expectation in terms of discrete random variable.

11. State the definition of Mathematical expectation using continuous random variable.

12. In a business venture a man can make a profit of ₹ 2,000 with a probability of 0.4 or have a loss of ₹ 1,000 with a probability of $0\cdot6$. What is his expected, variance and standard deviation of profit?

13. The number of miles an automobile tire lasts before it reaches a critical point in tread wear can be represented by a p.d.f.

$$f(x)=\begin{cases}\frac{1}{30}e^{-\frac{x}{30}}, & for\ x>0\\ 0, & for\ x\le 0\end{cases}$$

Find the expected number of miles (in thousands) a tire would last until it reaches the critical tread wear point.

14. A person tosses a coin and is to receive ₹ 4 for a head and is to pay ₹ 2 for a tail. Find the expectation and variance of his gains.

15. Let X be a random variable and $Y = 2X + 1$. What is the variance of Y if variance of X is 5 ?

Exercise 6.3

Choose the correct Answer

1. Value which is obtained by multiplying possible values of random variable with probability of occurrence and is equal to weighted average is called

(a) discrete value

(b) weighted value

(c) expected value

(d) cumulative value

RMHB3

2. Demand of products per day for three days are 21, 19, 22 units and their respective probabilities are $0\cdot29$, $0\cdot40$, $0\cdot35$. Pofit per unit is $0\cdot50$ paisa then expected profits for three days are

(a) 21, 19, 22 (b) 21.5, 19.5, 22.5

(c) 0.29, 0.40, 0.35 (d) 3.045, 3.8, 3.85

3. Probability which explains x is equal to or less than particular value is classified as

(a) discrete probability

(b) cumulative probability

(c) marginal probability

(d) continuous probability

4. Given $E(X) = 5$ and $E(Y) = -2$, then $E(X - Y)$ is

(a) 3 (b) 5 (c) 7 (d) −2

5. A variable that can assume any possible value between two points is called

(a) discrete random variable

(b) continuous random variable

(c) discrete sample space

(d) random variable

6. A formula or equation used to represent the probability distribution of a continuous random variable is called

(a) probability distribution

(b) distribution function

(c) probability density function

(d) mathematical expectation

7. If X is a discrete random variable and $p(x)$ is the probability of X, then the expected value of this random variable is equal to

(a) $\sum f(x)$ (b) $\sum[x+f(x)]$

(c) $\sum f(x)+x$ (d) $\sum xp(x)$

8. Which of the following is not possible in probability distribution?

(a) $\sum p(x) \geq 0$ (b) $\sum p(x)=1$

(c) $\sum x\, p(x)=2$ (d) $p(x)=-0.5$

9. If *c* is a constant, then $E(c)$ is

(a) 0 (b) 1

(c) $c\, f(c)$ (d) c

10. A discrete probability distribution may be represented by

(a) table

(b) graph

(c) mathematical equation

(d) all of these

11. A probability density function may be represented by

(a) table

(b) graph

(c) mathematical equation

(d) both (b) and (c)

12. If *c* is a constant in a continuous probability distribution, then $p(x=c)$ is always equal to

(a) zero (b) one

(c) negative (d) does not exist

13. $E[X-E(X)]$ is equal to

(a) $E(X)$ (b) $V(X)$

(c) 0 (d) $E(X)-X$

14. $E[X-E(X)]^2$ is

(a) $E(X)$ (b) $E(X^2)$

(c) $V(X)$ (d) $S.D(X)$

15. If the random variable takes negative values, then the negative values will have

(a) positive probabilities

(b) negative probabilities

(c) constant probabilities

(d) difficult to tell

16. If we have $f(x)=2x, 0 \leq x \leq 1$, then $f(x)$ is a

(a) probability distribution

(b) probability density function

(c) distribution function

(d) continuous random variable

17. $\int_{-\infty}^{\infty} f(x)dx$ is always equal to

(a) zero (b) one

(c) $E(X)$ (d) $f(x)+1$

18. A listing of all the outcomes of an experiment and the probability associated with each outcome is called

(a) probability distribution

(b) probability density function

(c) attributes

(d) distribution function

19. Which one is not an example of random experiment?

(a) A coin is tossed and the outcome is either a head or a tail

(b) A six-sided die is rolled

(c) Some number of persons will be admitted to a hospital emergency room during any hour.

(d) All medical insurance claims received by a company in a given year.

20. A set of numerical values assigned to a sample space is called

(a) random sample

(b) random variable

(c) random numbers

(d) random experiment

21. A variable which can assume finite or countably infinite number of values is known as

(a) continuous (b) discrete

(c) qualitative (d) none of them

22. The probability function of a random variable is defined as

$X=x$	-1	-2	0	1	2
$P(x)$	k	2k	3k	4k	5k

Then k is equal to

(a) zero (b) $\frac{1}{4}$

(c) $\frac{1}{15}$ (d) one

23. If $p(x)=\frac{1}{10}$, $x = 10$, then $E(X)$ is

(a) zero (b) $\frac{6}{8}$

(c) 1 (d) −1

24. A discrete probability function $p(x)$ is always

(a) non-negative (b) negative

(c) one (d) zero

25. In a discrete probability distribution the sum of all the probabilities is always equal to

(a) zero

(b) one

(c) minimum

(d) maximum

26. An expected value of a random variable is equal to it's

(a) variance

(b) standard deviation

(c) mean

(d) covariance

27. A discrete probability function $p(x)$ is always non-negative and always lies between

(a) 0 and ∞ (b) 0 and 1

(c) −1 and +1 (d) $-\infty$ and $+\infty$

28. The probability density function $p(x)$ cannot exceed

(a) zero (b) one

(c) mean (d) infinity

29. The height of persons in a country is a random variable of the type

(a) discrete random variable

(b) continuous random variable

(c) both (a) and (b)

(d) neither (a) nor (b)

30. The distribution function $F(x)$ is equal to

(a) $P(X=x)$

(b) $P(X\leq x)$

(c) $P(X\geq x)$

(d) all of these

Miscellaneous Problems

1. The probability function of a random variable X is given by

$$p(x)=\begin{cases}\frac{1}{4}, & for\ x=-2\\ \frac{1}{4}, & for\ x=0\\ \frac{1}{2}, & for\ x=10\\ 0, & elsewhere\end{cases}$$

Evaluate the following probabilities.

(i) $P(X\leq 0)$ (ii) $P(X<0)$

(iii) $P(|X|\leq 2)$ (iv) $P(0\leq X\leq 10)$

2. Let X be a random variable with cumulative distribution function

$$F(x)=\begin{cases}0, \text{ if } x<0\\ \frac{x}{8}, \text{ if } 0\le x<1\\ \frac{1}{4}+\frac{x}{8}, \text{ if } 1\le x<2\\ \frac{3}{4}+\frac{x}{12}, \text{ if } 2\le x<3\\ 1, \text{ for } 3\le x.\end{cases}$$

(a) Compute: (i) $P(1\le X\le 2)$ and (ii) $P(X=3)$.

(b) Is X a discrete random variable? Justify your answer.

3. The p.d.f. of X is defined as

$$f(x)=\begin{cases}k, & \text{for } 0<x\le 4\\ 0, & \text{otherwise}\end{cases}$$

Find the value of k and also find $P(2\le X\le 4)$.

4. The probability distribution function of a discrete random variable X is

$$f(x)=\begin{cases}2k, & x=1\\ 3k, & x=3\\ 4k, & x=5\\ 0, & \text{otherwise}\end{cases}$$

where k is some constant. Find (a) k and (b) $P(X>2)$.

5. The probability density function of a continuous random variable X is

$$f(x)=\begin{cases}a+bx^2, 0\le x\le 1;\\ 0, \text{ otherwise.}\end{cases}$$

where a and b are some constants. Find (i) a and b if $E(X)=\frac{3}{5}$ (ii) $Var(X)$.

6. Prove that if $E(X)=0$, then $V(X)=E(X^2)$.

7. What is the expected value of a game that works as follows: I flip a coin and, if tails you pay ₹ 2; if heads you pay ₹ 1. In either case I also pay you ₹ 0.50

8. Prove that, (i) $V(aX)=a^2V(X)$ and (ii) $V(X+b)=V(X)$

9. Consider a random variable X with p.d.f

$$f(x)=\begin{cases}3x^2, \text{ if } 0<x<1\\ 0, \text{otherwise}\end{cases}$$

Find $E(X)$ and $V(3X-2)$.

10. The time to failure in thousands of hours of an important piece of electronic equipment used in a manufactured DVD player has the density function

$$f(x)=\begin{cases}2e^{-2x}, & x>0\\ 0, & \text{otherwise}\end{cases}$$

Find the expected life of this piece of equipment.

Summary

- A variable which can assume finite number of possible values or an infinite sequence of countable real numbers is called a discrete random variable.
- Probability mass function (p.m.f.)

$$P_X(x)=p(x)=\begin{cases}P(X=x_i)=p_i=p(x_i) & \text{if } x=x_i, i=1,2,\ldots,n,\ldots\\ 0 & \text{if } x\ne x_i\end{cases}$$

Conditions:

- $p(x_i)\ge 0 \;\forall i$ and
- $\sum_{i=1}^{\infty} p(x_i)=1$

- Discrete distribution function (d.f.):

  $F_X(x) = P(X \le x)$ *for all* $x \in R$

  $i.e., F_X(x) = \sum_{x_i \le x} p(x_i)$

- A random variable X which can take on any value (integral as well as fraction) in the interval is called continuous random variable.

- Probability density function (p.d.f.)

  The probability that a random variable X takes a value in the (open or closed) interval $[t_1, t_2]$ is given by the integral of a function called the probability density function $f_X(x)$

  $$P(t_1 \le X \le t_2) = \int_{t_1}^{t_2} f_X(x)dx$$

  Other names that are used instead of probability density function include density function, continuous probability function, integrating density function.

Conditions:

- $f(x) \ge 0 \;\forall\; x$
- $\int_{-\infty}^{\infty} f(x)dx = 1$

- Continuous distribution function

  If X is a continuous random variable with the probability density function $f_X(x)$, then the function $F_X(x)$ is defined by

  $$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f_X(t)dt, -\infty < x < \infty$$

  is called the distribution function (d.f) or sometimes the cumulative distribution function (c.d.f) of the random variable X.

- Properties of cumulative distribution function (c.d.f.)

  The function $F_X(X)$ or simply $F(X)$ has the following properties

  (i) $0 \le F(x) \le 1, -\infty < x < \infty$

  (ii) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$, and $F(+\infty) = \lim_{x \to \infty} F(x) = 1$.

  (iii) $F(\cdot)$ is a monotone, non-decreasing function; that is, $F(a) \le F(b)$ for $a < b$.

  (iv) $F(\cdot)$ is continuous from the right; that is, $\lim_{h \to 0} F(x+h) = F(x)$.

  (v) $F'(x) = \frac{d}{dx}F(x) = f(x) \ge 0$

(vi) $F'(x) = \frac{d}{dx}F(x) = f(x) \Rightarrow dF(x) = f(x)dx$

(vii) $dF(x)$ is known as probability differential of X.

(viii) $P(a \le x \le b) = \int_a^b f(x)dx = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$

$= P(X \le b) - P(X \le a)$

$= F(b) - F(a)$

- Mathematical Expectation

  The expected value is a weighted average of the values of a random variable may assume.

- Discrete random variable with probability mass function (p.m.f.)

  $E(X) = \sum_x x\,p(x)$

- Continuous random variable with probability density function

  $E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

- The mean or expected value of X, denoted by φ_X or E(X).
- The variance is a weighted average of the squared deviations of a random variable from its mean.
- $Var(X) = \sum \left[x - E(X)\right]^2 p(x)$

  if X is discrete random variable with probability mass function $p(x)$.

- $Var(X) = \int_{-\infty}^{\infty} \left[X - E(X)\right]^2 f_X(x)\,dx$

  if X is continuous random variable with probability density function $f_X(x)$.

- Expected value of $[X - E(X)]^2$ is called the variance of the random variable.

  i.e., $Var(X) = E\left[X - E(X)\right]^2 = E\left(X^2\right) - \left[E(X)\right]^2$

  where $E\left(X^2\right) = \begin{cases} \sum_x x^2\,p(x), \text{ if } X \text{ is Discrete Random Variable} \\ \int_{-\infty}^{\infty} x^2 f(x)\,dx, \text{ if } X \text{ is Continuous Random Variable} \end{cases}$

- If X is a random variable, the standard deviation of X, denoted by σ_X, is defined as $+\sqrt{Var\left[X\right]}$.
- The variance of X, denoted by $\sigma_X{}^2$ or $Var(X)$ or $V(X)$.
- Properties of Mathematical expectation

  (i) $E(a) = a$, where 'a' is a constant

  (ii) $E(aX) = aE(X)$

  (iii) $E(aX + b) = aE(X) + b$, where 'a' and 'b' are constants.

(iv) If $X \geq 0, then\, E(X) \geq 0$

(v) $V(a) = 0$

(vi) If X is random variable, then $V(aX+b) = a^2 V(X)$

- Raw moments

$$\varphi'_r = E\left(X^r\right) = \begin{cases} \sum_x x^r p(x), & for\ discrete \\ \int_{-\infty}^{\infty} x^r f(x)dx, & for\ continuous \end{cases}$$

- Central Moments

$\varphi_r = E[(X - \varphi_X)^r]$

$\varphi'_1 = E(X) = \varphi_X$, the mean of X.

$\varphi_1 = E[X - \varphi_X] = 0$.

$\varphi_2 = E[(X - \varphi_X)^2]$, the variance of X.

GLOSSARY (கலைச்சொற்கள்)	
Absolutely Convergent	முற்றிலும் குவிதல் இயல்புடைய
Biased	நடுநிலையற்ற
Central moments	மையநிலை விலக்கப் பெருக்கம்
Continuous distribution function	தொடர்ச்சியான பரவல் சார்பு
Continuous random variable	தொடர்ச்சியான சமவாய்ப்பு மாறி
Cumulative	திரள், குவிந்த
Discrete distribution function	தொடர்ச்சியற்ற பரவல் சார்பு
Discrete random variable	தொடர்ச்சியற்ற சமவாய்ப்பு மாறி
Distribution function	பரவல் சார்பு
Event	நிகழ்வு, நிகழ்ச்சி
Expectation	எதிர்பார்த்தல்
Expected value	எதிர்பார்க்கத்தக்க மதிப்பு / எதிர்பார்த்தல் மதிப்பு
Mathematical expectation	கணக்கியல் எதிர்பார்த்தல்
Mean	சராசரி
Moments	விலக்கப் பெருக்கங்கள்
Probability function	நிகழ்தகவுச் சார்பு
Probability mass function	நிகழ்தகவு நிறைச் சார்பு
Random variable	சமவாய்ப்பு மாறி
Standard deviation	திட்ட விலக்கம், நியமச்சாய்வு
Unbiased	நடுநிலையான
Urn	குடுவை, கலசம்
Variance	மாறுபாட்டு அளவை / பரவற்படி
Weighted average	நிறைச் சராசரி