14. A mass of 1 kg attached to the bottom of a spring has certain frequency of vibration. What mass is to be added in order to reduce the frequency by half ?

(A) 1 kg (B) 2 kg (C) 3 kg (D) 4 kg

15. A pendulum suspended from the ceiling of a compartment of a train has periodic time 2 s. When the train is accelerating at 10 ms^{-2}. What will be its time period when the train retards at 10 m s^{-2}.

(A) 2 s (B) $\sqrt{2}$ s (C) $2\sqrt{2}$ s (D) $\frac{2}{\sqrt{2}}$ s

ANSWERS

1. (B)	2. (A)	3. (D)	4. (C)	5. (B)	6. (C)
7. (C)	8. (A)	9. (C)	10. (B)	11. (B)	12. (D)
13. (D)	14. (C)	15. (A)			

Answer the following questions in short :

1. What is the work done by simple pendulum in one complete oscillation ?
2. What is the periodic time of a pendulum in freely falling lift ?
3. Write equation for peiodic time of oscillation of the liquid in U-tube.
4. What is an epoch ? In which unit it is measured ?
5. Amplitude of an SHO is 4 cm. At what distance from the equilibrium position, the kinetic energy and potential energy becomes equal ?
6. What is the SI unit of force constant ?
7. Write the relation between the acceleration amplitude (a), the displacement amplitude (A) and the angular frequency (ω) for SHM.
8. Why does a simple pendulum eventually stop ?
9. Write expression of the mechanical energy of damped harmonic oscillation for $b << \sqrt{km}$.
10. Write general form of the second order differential equation for forced oscillation.

Answer the following questions :

1. Define periodic motion and oscillatory motion. Give proper examples of it.
2. Deduce an expression for the time period of a simple pendulum.
3. What are damped oscillations ? What are the factors affecting its motion ?
4. Deduce the relation for the total energy of damped harmonic oscillator.
5. Explain forced oscillations and resonance.
6. Show that for a particle in linear SHM the average KE over a period of oscillation equals the average PE over the same period.

7. Obtain the co-ordinates of the points where the KE and PE against displacement graphs intersect.

8. What is the nature of acceleration against displacement curve of SHM ? What is the slope of this curve ?

9. Periodic time of the particle excluding SHM is $T = 2\pi\sqrt{\frac{m}{k}}$. Explain why then the periodic time of a simple pendulum is independent of a mass of the pendulum ?

10. What provides the restoring force for simple harmonic oscillator in the following cases ?

(i) Simple pendulum (ii) Spring (iii) Column of mercury in U-tube.

Solve the following problems :

1. Obtain the equation for SHM of the Y-projection of the radius vector of the revolving particle P in case (*a*) and (*b*) of Figure 7.22.

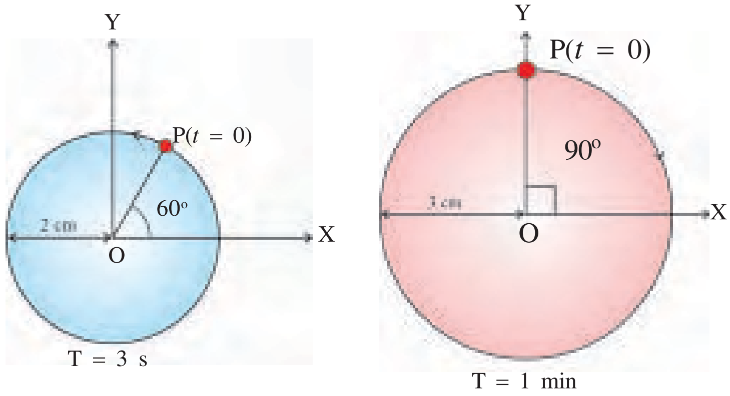

Figure 7.22

[**Ans. :** (*a*) $y = 2\sin\left(\frac{2\pi t}{3} + \frac{\pi}{3}\right)$ (*b*) $y = 3\cos\left(\frac{\pi}{30}t\right)$]

2. Three springs are connected to a mass m = 80 g as shown in Figure 7.23. What is the effective spring content and periodic time, if $k = 2$ N m^{-1} ?

[**Ans.** : k = 8 Nm^{-1}, T = 0.628 s]

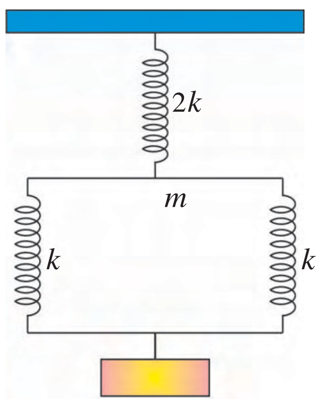

Figure 7.23

3. A spring of length l and force constant k is cut into two parts of length l_1 and l_2. Here $l_1 = nl_2$. Obtain force constants k_1 and k_2 respectively of these parts in terms of n and k. [**Ans. :** $k_1 = \left(1 + \frac{1}{n}\right)k$, $k_2 = (n+1)k$]

4. An oscillator of mass 100 g is performing damped oscillations. Its amplitude becomes half of its initial amplitude after 100 oscillations. If its period is 2 s find the damping co-efficient. [**Ans. :** 0.693 *dyn* s cm^{-1}]

5. Amplitude of an SHO is A. When it is at a distance y from the mean position of the path of its oscillation, the SHO receives blow in the direction of its motion which doubles its velocity instantaneously. Find the new amplitude of its oscillations. [**Ans. :** $\sqrt{4A^2 - 3y^2}$]

6. For an SHM prove that $a^2T^2 + 4\pi^2v^2$ = constant, where a and v are acceleartion and velocity respectively at any instant. T is periodic time.

7. A simple pendulum has a length L and a bob of mass m. The bob is oscillating with amplitude A. Show that the maximum tension (T) in the string is (for small angular displacement). $T_{max} = mg\left[1 + \left(\frac{A}{L}\right)^2\right]$.

8. Two simple harmonic motions are represented by $y_1 = 10 \sin\frac{\pi}{4}(12t + 1)$ and $y_2 = 5(\sin 3\pi t + \sqrt{3}\cos 3\pi t)$. Find out the ratio of their amplitudes. What are the time period of two motions ? [**Ans. :** $\frac{A_1}{A_2} = 1,\ T_1 = T_2 = \frac{2}{3}$ s]

9. For a linear harmonic oscillator the force constant is 2×10^6 N/m and total mechanical energy is 160 J. At some instant of time, its displacement is 0.01 m. Find its potential energy and kinetic energy at this position.

[**Ans. :** 100 J, 60 J]

10. For a linear SHM, when the distance of the oscillator from the equilibrium postion has values y_1 and y_2, the velocities are v_1 and v_2. Show that the time period of oscillation is $T = 2\pi\left[\frac{y_2^2 - y_1^2}{v_1^2 - v_2^2}\right]^{\frac{1}{2}}$.

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CHAPTER 8

WAVES

8.1 Introduction

Friends, earlier we have studied that the universe is made up of matter and radiation. This radiation propagates in the form of waves. Waves have basic importance in almost every branch of physics. Light and sound energy also propagate in the form of waves. Different types of radiant energy emitting from the sun also reaches us in the form of waves. Music produced from musical instruments also reaches us in the form of waves. Communication done through radio, television and mobile is due to the waves. In 20th century, concept of matter wave was introduced due to which importance of the waves also increase.

In this chapter we will learn about waves, types of waves, speed of waves in different medium, reflection of waves, superposition of waves, beats and Doppler effect.

8.2 Waves

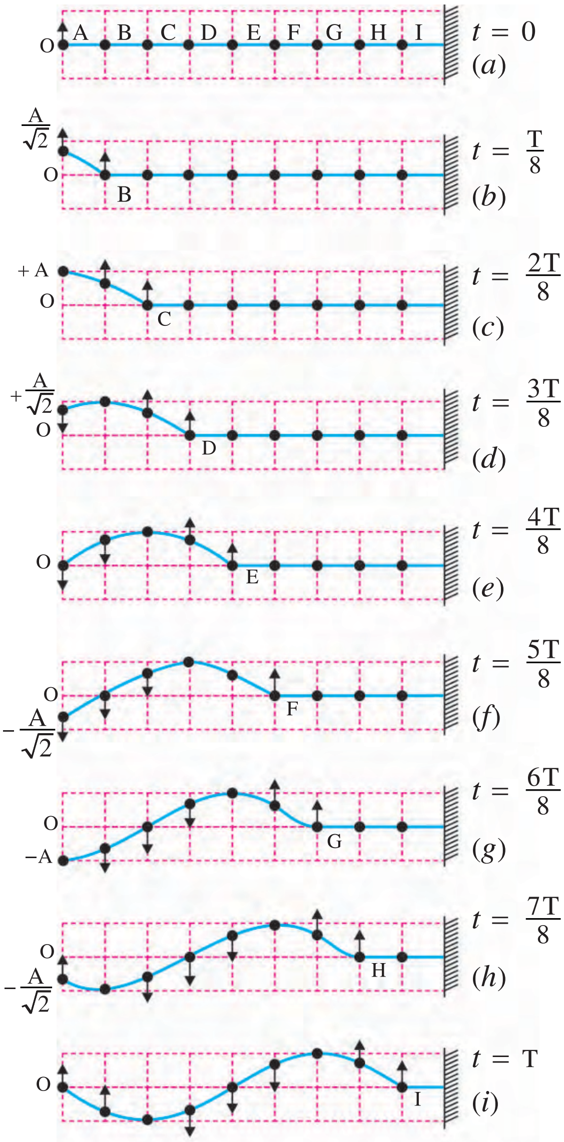
When a particle moves in space it carries the kinetic energy associated with it. There is another way to transport energy in which the particle oscillates near its position and yet the energy reaches too far from it. They transport their associated energy to the far distance without leaving their position. Sound is transmitted in air in this manner. When you say 'Hello' to your friend, no material particle is ejected from your lips and reaches to your friend's ear. You create some disturbance in the air close to your lips which propagates as a wave and reaches to ear of your friend.

To understand clearly the concept of a wave, consider a long and elastic string with one end fixed to rigid support and other held by a person. The person pulls on the string keeping it tight. Here, string is a one dimensional elastic medium. As shown in Figure 8.1, suppose that A, B, C, ... I are the particles of a string. At time $t = 0$ all the particles of the medium are in the equilibrium state. (See Figure 8.1*a*)

(i) Suppose, at $t = 0$ a disturbance is produced by the person in the particle A so that it starts simple periodic motion according to $y = A \sin\omega t$. The period of this oscillation is T.

(ii) Because of elastic property of the medium, suppose the effect of disturbance produced at A is transmitted to particle B in time $\frac{T}{8}$. During this time $\frac{T}{8}$, the displacement of a particle A would be $y = A\sin\left(\frac{2\pi}{T}\right)\left(\frac{T}{8}\right) = \frac{A}{\sqrt{2}}$ and particle B is on the verge of starting its simple periodic oscillation, (See Figure 8.1 *b*)

(iii) Now, during an additional time period of $\frac{T}{8}$, that is at $\frac{T}{8} + \frac{T}{8} = \frac{T}{4}$, the disturbance produced at A reaches C and at that moment C is on the verge of starting its oscillation. During this time $\frac{T}{4}$, the displacement of particle A would be $y = A\sin\left(\frac{2\pi}{T}\right)\left(\frac{T}{4}\right) = A$ i.e. Displacement of A is equal to its amplitude and that of B is equal to $\frac{A}{\sqrt{2}}$. (See Figure 8.1(*c*)).



Wave generation on a string

Figure 8.1

(iv) Thus, due to the disturbance produced at A, the subsequent particles start oscillations and transmit the effect of their oscillation to the subsequent particles and the disturbance propagates in the medium.

(v) In this way the propagating disturbance reaches the particle D at $t = \frac{3T}{8}$, the particle E at $\frac{4T}{8}$ ...... and the particle I at time T. At time T one oscillation of A is completed and the particle I is just to start its oscillation.

This entire situation is shown in Figure 8.1. Remember that the particles of the medium were in the equilibrium position. At time $t = 0$, we produced a simple periodic disturbance at the particle A. This disturbance has travelled in the medium and reached the particle I at time $t = T$.

(vi) Here, disturbance considered is such that it produces a simple periodic motion in the particle A and hence the shape produced in the string is like a sine curve. If the displacement or oscillation of particle A had been of some other type, the shape formed in the string would also have been accordingly of the other type. **Thus, the shape formed in the string gives an idea of the type of disturbance.** For example, if the free end of the string snappad one, then the shape formed in string is shown in Figure 8.2, which is known as a pulse.

Shape formed on the string according to the type of disturbance

Figure 8.2

As the time elapses this disturbance (or shape in the string) passes over the particles J, K, L,..... etc. Here, the shape is that of the sine curve is lying between the particles A and I at time $t = T$. This shape proceeds further along the string and comes between particles I and Q at time $t = 2T$ as shown in Figure 8.3. During this time the particles between A and I stop oscillating and string comes back to its original position.

Shape of the string at t = 2T

Figure 8.3

Thus, by producing a disturbance at any particle in the string, a shape corresponding to the disturbance is produced and that shape (without alteration) moves along the string, which means that the disturbance propagates in the medium of the string. **The motion of the disturbance in the medium (or in free space) is called a wave disturbance or generally a wave.**

Remember that here the particles of the string A, B, C... are not moving as a 'single unit' in the medium but they only displace or oscillate about their equilibrium positions. Thus, wave is not physical 'body' travelling in the medium. As the effect of disturbance produced in any part of the medium is being experienced by the subsequent particles of the medium, the wave is said to propagate. **After the disturbance has passed through any particle, it comes back to its equilibrium position.**

When the engine of a railway train joins the coaches, in the beginning the first coach vibrates, then the second coach and then the third coach and so on. Thus, the effect of vibration moves from the first to the last coach. This phenomenon is the propagation of the wave in the medium 'made up of railway coaches.'

Wavetrain

In the above discussion, if the particle continues to oscillate in its simple harmonic motion, the second waveform generated after the first one follows it and so on. Thus, a series of waveform appears to move ahead as a continuous chain. Such a series of propagating waveform is called a wave train.

We discussed a situation in which particles participating in wavemotion are executing a simple harmonic oscillation, here the wave shape formed in medium is of the nature of a sine (or equivalent cosine) curve. Such a wave is called a **harmonic wave.**

If the waves are continuously moving ahead in the medium, they are called **travelling or progressive waves.**

8.3 Classification of Waves

(i) Mechanical waves : The waves which require elastic medium for their transmission are called mechanical waves. Such a wave propagates due to the elastic properties of the medium. For example, waves on a string, ripples on the water suface, sound waves and seismic waves. All these waves have the characteristics that they are governed by Newton's laws.

(ii) Electromagnetic waves : For the propagation of electromagnetic waves no material medium is essential. They can propagate in the vacuum also. In this types of waves the disturbance in the electric and magnetic fields that propagates. Here, instead of particles, the vectors of the electric and magnetic field intensities are oscillating.

Light waves, radio waves, microwave, X−ray etc. are the examples of the electromagnetic waves. (More information you will get in Std. 12)

(iii) Matter waves : Matter waves are associated with moving electrons, protons, neutrons and other fundamental particles and even atoms and molecules. These particles constituting matter, therefore, such waves are called matter waves. The concept of these types of wave you will learn in Std. 12. From the concept of these waves scientific instruments are developed in modern technology. The matter waves associated with electron are employed in the electron microscope.

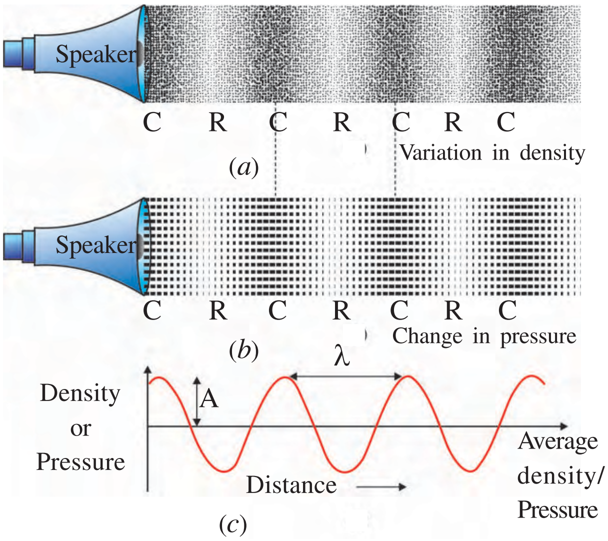
In this chapter we will study only the mechanical waves.

Transverse wave : Waves in which the oscillations of the particles are in a direction perpendicular to the direction of wave propagation is called the **transverse wave.** The waves along a string discussed in article 8.2 is an example of transverse wave. Electromagnetic waves (e.g. light waves) are also a transverse waves. In such waves the locations of the maximum displacement of the particle in one direction are called the **'crests'** and locations of maximum displacement in the opposite direction are called **'trough'**.

These waves propagate through a medium in the form of crests and troughs.

Longitudinal wave : Waves in which the oscillations of the particles of medium are along the direction of wave propagation are called **longitudinal waves.** Sound waves propagating in air are longitudinal. Such waves propagate through a medium by forming condensations and rarefaction in the medium. When waves propagate in medium, the particles at medium oscillate about their equilibrium position, in the direction of propagation of waves.

For simplicity, the positions of the particles of air at some instant of time in case of longitudinal waves is shown in Figure 8.4.



Longitudinal wave in air

Figure 8.4

When sound waves pass through that region of air, the air molecules in certain region are pushed very close to each other during their oscillations. Hence, both density and pressure of air increase in such regions. In such region **condensation** is said to be formed. In the regions between consecutive condensations, the air molecules are found to be quite separated. In such regions density and pressure of air decreases and here **rarefaction** is said to be formed. (See Figure 8.4)

Thus, during the propagation of sound the layers of medium perform oscillation about their mean positions and during this the condensations and rarefactions are alternatingly formed. As the effect of such oscillations reaches one layer after the other, the condensations and rarefactions propagate further and further in the medium. In this way the sound propagates in a medium. During the propagation of sound the pressure in different region of the medium changes with time and position. Hence, such waves are also called the **pressure waves.**

The direction of the oscillations of the particles of the medium is perpendicular to the direction of propagation of transverse waves in the medium. Hence during the propagation of the transverse waves every element of the medium experience shearing strain. But shearing stress is possible only in solid medium. So, the transverse waves can propagate in solid medium like string, wire, rod but they cannot propagate in a fluid medium.

During the propagation of the longitudinal waves, the oscillations of the particles of the medium are in the direction of propagation of the waves. Hence, compressive strain is produced during the propagation of these waves. Now, compressive stress is possible in solids, liquids and gases. So, the longitudinal waves can propagate in any medium.

Thus, in a solid medium both types of mechanical waves, transverse waves and longitudinal waves can propagate whereas in a fluid medium only longitudinal waves can propagate.

[During an earthquake two types of waves, transverse and longitudinal are produced on the earth. They are known as S-wave (secondary wave) and P-wave (primary wave) respectively. Longitudinal wave (P-wave) is similar to sound waves produced in the earth's interior. The speed of P-wave is approximately 4 – 8 km/s and that of S-wave is approximately 2 – 5 km/s. In an S-wave, particles in the earth's interior vibarate at right angles to the direction of the wave propagation. By measuring the time interval between the first arrivals of P-wave and S-wave, the origin of earthquake (epicentre) can be determined.

8.4 Amplitude of A Wave, Propagation of Energy in a Wave, Wavelength And Frequency

Amplitude of a wave :

Amplitude of wave is the amplitude of

oscillation of particle of the medium. As shown in Figure 8.5 amplitude of the wave is A.

Propagation of energy in a wave :

A particle has to be displaced from its mean position in order to produce a wave. Hence some work has to be done on the particle. This energy imparted to the particle will be in the form of kinetic and potential energy of its oscillations. As the successive particles experience the disturbance, this energy is communicated to them. Thus, energy is propagated in a wave. If the medium has some internal friction, energy is dissipated in the form of heat and hence, the wave weakens on propagation.

Energy passing through a unit area taken in the direction normal to the propagation of the wave in one second is called intensity of wave.

$$\text{Wave Intensity (I)} = \frac{\text{Energy / Time}}{\text{Area}}$$

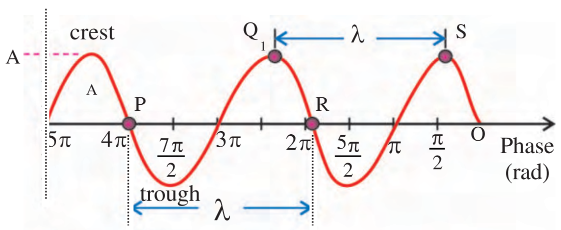
SI unit of intensity of wave is $\frac{\text{J/s}}{\text{m}^2}$ or $\frac{\text{W}}{\text{m}^2}$.

Its dimensional formula is $M^1L^0T^{-3}$.

Energy of an oscillatory particle is $E = \frac{1}{2}kA^2$, hence the intensity of wave is directly proportional to the square of its amplitude. ($I \propto A^2$).

Wavelength :

The linear distance between any two points or particles having phase difference of 2π rad is called the wavelength (λ) of the wave. Its SI unit is m.



Amplitude and wavelength of a wave

Figure 8.5

As shown in Figure 8.5 the phase difference of oscillation between particles P and R is $4\pi - 2\pi = 2\pi$ rad. Hence, the distance between P and R represent the wavelength (λ) of a wave. From the figure it is clear that phase difference between consecutive crests or consecutive trough is 2π rad. Therefore, the distance between consecutive crests/trough is also a wavelength of a wave. Same way, in case of the sound waves the distance between consecutive condensations or consecutive rarefactions also represents the wavelength.

Wave number and wave vector :

Number of waves per unit distance is called wave number $\left(\frac{1}{\lambda}\right)$. The SI unit of wave number is m^{-1}.

In the wave propagation the particles at a distance of λ has the phase difference of 2π rad. Hence, the particle at a unit distance has phase difference of $\frac{2\pi}{\lambda}$. $\frac{2\pi}{\lambda}$ is known as wave vector or angular wave number or propagation constant (k).

$$k = \frac{2\pi}{\lambda}$$

The SI unit of k is rad/m. Its dimensional formula is $M^0L^{-1}T^0$. Wave vector is in the direction of wave propagation.

Frequency of a wave :

The number of oscillations performed by the particle of medium in one second is known as the frequency of oscillation of particle. Frequency (f) of the wave is just the frequency of oscillation of the particles of the medium. The number of the wave passing through point in one second is called frequency of the wave.

Its SI unit is s^{-1} or Hz (Hertz).

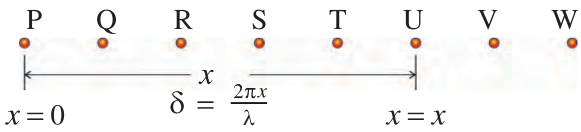
$\omega = 2\pi f$ is called angular frequenecy of the wave. $T = \frac{1}{f}$ is the periodic time of the wave.

8.5 Wave Equation

A complete description for a wave propagation can be obtained if we know displacements of all the particles of medium participating in the wave motion at any time. For this purpose we shall derive the wave equation for wave in one dimension, which gives displacement of a particle having coordinate x at time t. From such an equation, substituting the appropriate values for x and t, we get the displacement for any particle at a required time and thus obtain a description of the wave motion.

Such an equation is called **wave equation.** (Here, we shall discuss wave equation only for one dimension).

Here we shall obtain an equation for travelling wave or progressive harmonic waves. To obtain the wave equation of a wave propagating in positive x direction, consider particles of a medium as shown in Figure 8.6.



Wave equation

Figure 8.6

Suppose, at $t = 0$, simple harmonic oscillations of the particle P start with zero intial phase i.e. wave originates at P at time $t = 0$. The x-coordinate of the particle P is zero as well as initial phase is also zero ($\phi = 0$). The equation for the displacement of this particle would be,

$$y = A \sin\omega t \qquad (8.5.1)$$

Now when the wave originating at P travels through a distance x, the medium particle (U) lying at a distance x from P starts its simple harmonic motion and the phase of its oscillation would be less than that of P. Let the phase of this particle (U) be δ less than that of P. Hence the equation for the displacement of particle at distance x from P, would be,

$$y = A \sin(\omega t - \delta) \qquad (8.5.2)$$

Let the wavelength of wave be λ. We know that the phase of the particle at a distance λ from P, is less than that of P by 2π. Hence, the phase of the particle at a distance x from P would be less than that of P by $\frac{2\pi x}{\lambda}$.

$$\therefore \ \delta = \frac{2\pi x}{\lambda} \qquad (8.5.3)$$

Substituting δ in equation (8.5.2)

$$y = A \sin\left(\omega t - \frac{2\pi x}{\lambda}\right)$$

But $\frac{2\pi}{\lambda} = k$

$$\therefore \ y = A \sin(\omega t - kx) \qquad (8.5.4)$$

Here, $(\omega t - kx)$ is known as the phase of the wave at distance x from the origin at time t. The direction of wave vector k is taken along the direction of propagation of the wave.

Equation (8.5.4) is the wave equation for the progressive harmonic wave travelling in the direction of the increasing value of x. **If the wave is travelling in the direction of decreasing value of x then $\omega t - kx$ is replaced by $\omega t + kx$.**

$$y = A \sin(\omega t + kx) \qquad (8.5.5)$$

substituting $\omega = \frac{2\pi}{T}$ and $k = \frac{2\pi}{\lambda}$ in equation (8.5.4)

$$y = A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad (8.5.6)$$

If the velocity of wave is v, then substituting $\lambda = vT$ in above equation

$$y = A \sin 2\pi\left(\frac{t}{T} - \frac{x}{vT}\right)$$

$$y = A \sin 2\pi f\left(t - \frac{x}{v}\right) \quad \left(\because \ \frac{1}{T} = f\right) \qquad (8.5.7)$$

Now,

$$y = A \sin 2\pi \frac{f}{v}(vt - x)$$

$$\therefore \ y = A \sin \frac{2\pi}{\lambda}(vt - x) \quad (\because \ v = f\lambda)$$

$$(8.5.8)$$

The above equations (8.5.6), (8.5.7) and (8.5.8) are the different forms of wave equation for the progressive harmonic wave.

If particle P is oscillating with initial phase ϕ, then the wave equation (8.5.4) will be as follows :

$$y = A \sin(\omega t - kx + \phi) \qquad (8.5.9)$$

8.6 Wave speed and phase speed

Wave travells a distance λ in periodic time T.

$$\therefore \ \text{wave speed } v = \frac{\text{Distance}}{\text{Time}} = \frac{\lambda}{T}$$

But $\frac{1}{T} = f$

$$\therefore \ v = f\lambda \qquad (8.6.1)$$

$$= \frac{\lambda(2\pi f)}{2\pi}$$

But, $2\pi f = \omega$ and $\frac{2\pi}{\lambda} = k$

$$\therefore \ v = \frac{\omega}{k} \qquad (8.6.2)$$

So far in the discussion of wave motion we have seen that amplitude, period of oscillation and frequency of oscillation (f) of the particles of medium are the amplitude, periodic time and frequency of the wave respectively.

But the velocity of oscillating particle and velocity of wave are not the same.

Note that, **the frequency of the wave is the property of the source of the wave while the wavelength is the property of the medium in which the wave propagates.**

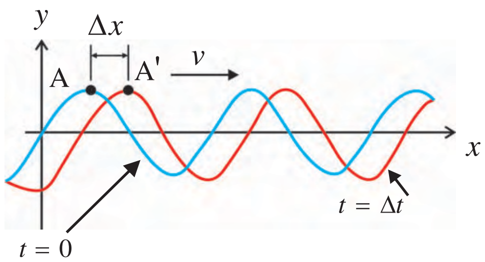
In the different mediums the wave speed is different. Therefore, the wavelength of the wave is also different in different type of medium. But in a given medium the wave speed is constant.

Phase Speed

As shown in Figure 8.7 the wave is travelling in the direction of increasing value of x. The entire wave pattern is moving a distance Δx in that direction during the interval Δt. As the wave moves, each point of the moving wave form (such as point A) retains its displacement. (Remember that points on the string do not retain their displacement but point on the wave forms do). For each point on the wave pattern phase must be constant. In Figure 8.7 phase at point A and A' is same.

$$\therefore \ \omega t - kx = \text{constant} \qquad (8.6.3)$$

Here, both x and t are changing. As t increases, x must also increase to keep the $\omega t - kx$ constant. This confirms that the wave pattern is moving towards increasing x.



Wave motion

Figure 8.7

Differentiating above equation with respect to t.

$$\frac{d}{dt}(\omega t - kx) = 0$$

$$\therefore \ \omega - k\frac{dx}{dt} = 0$$

$$\therefore \ \frac{dx}{dt} = v = \frac{\omega}{k} \qquad (8.6.4)$$

Here, v is the phase speed of the wave.

Above equation (8.6.4) is similar to the equation (8.6.2). So, the wave speed which we find is the phase-speed of the waves in reality.

Illustration 1 : The frequency of the radio-waves broadcast by Ahmedabad Vividhbharati is 96.7 MHz. Find the wavelength, wave vector and angular frequency of these waves. Speed of radio waves in air is 3×10^8 m/s.

Solution :

$f = 96.7$ MHz $= 96.7 \times 10^6$ Hz

$v = 3 \times 10^8$ m/s

Wave speed, $v = f\lambda$

$$\therefore \ \lambda = \frac{v}{f} = \frac{3 \times 10^8}{96.7 \times 10^6} = 3.102 \text{ m}$$

$$\text{Wave vector, } k = \frac{2\pi}{\lambda} = \frac{2 \times 3.14}{3.102} = 2.024 \text{ rad/m}$$

Angular frequency, $\omega = 2\pi f$

$= 2 \times 3.14 \times 96.7 \times 10^6$

$= 6.07 \times 10^8$ rad/s

Illustration 2 : The wave equation of a propagating wave is given by $y = 0.5\sin(x - 60t)$ cm. Find, (*i*) amplitude of the wave (*ii*) wave vector (*iii*) wavelength (*iv*) angular frequency and frequency of wave (*v*) periodic time and (*vi*) wave speed.

Solution : Compare the equation

$y = 0.5 \sin(x - 60t) = -0.5\sin(60t - x)$

with wave equation

$y = A \sin(\omega t - kx)$

(i) Amplitude of a wave A $= -0.5$ cm

(ii) wave vector $k = 1$ rad/cm

$$\text{(iii) wavelength } \lambda = \frac{2\pi}{k} = \frac{2 \times 3.14}{1} = 6.28 \text{ cm}$$

(iv) Angular frequency of a wave $\omega = 60$ rad/s

Now, from $\omega = 2\pi f$, the frequency of wave,

$$f = \frac{\omega}{2\pi} = \frac{60}{2 \times 3.14} = 9.55 \text{ Hz}$$

(v) Periodic time $T = \frac{1}{f} = \frac{1}{9.55} = 0.105$ s

(vi) Wave speed $v = \frac{\omega}{k} = \frac{60}{1} = 60$ cm/s

Illustration 3 : How far does sound travel in air when a tuning fork of frequency 250 Hz completes 50 vibrations ? The speed of sound in air is 340 m/s.

Solution : Wavelength of the wave produced from tuning fork $\lambda = \frac{v}{f} = \frac{340}{250} = 1.36$ m.

One wavelength is the distance travelled by the wave in one complete vibration of tuning fork.

$\therefore$ Distance travelled by the sound in 50 vibrations.

$= 50 \times \lambda$

$= 50 \times 1.36 = 68$ m

Illustration 4 : A stone dropped from the top of a tower of height 100 m high splashed into the water of a pond near the tower. When is the splash heard at the top ? The speed of sound in air is 340 m/s. At what time the splash is heared at the top, after it is dropped ?

Solution : Suppose t_1 is the time taken by the stone to reach the surface of water and t_2 is the time taken by the splash to reach from water surface to the top. The splash will be heard at the top of the tower after time $t = t_1 + t_2$.

Now, time t_1 taken by stone to reach water surface can be determine as follows :

$s = v_0 t_1 + \frac{1}{2} g\, t_1^2$

$s = 100$ m, $v_0 = 0$, $g = 9.8$ m/s^2

$\therefore\ 100 = 0 + \frac{1}{2}(9.8)t_1^2$

$\therefore\ t_1 = 4.52$ s

Now, time t_2 taken by the splash to reach water surface to the top is,

$t_2 = \frac{\text{Distance}}{\text{Sound speed}} = \frac{100}{340} = 0.29$ s

$\therefore\ t = t_1 + t_2 = 4.52 + 0.29 = 4.81$ s

Illustration 5 : Equation of a one dimensional propagating wave is,

$y = 5\sin 30\pi \left(t - \frac{x}{240} \right)$.

Here, y is in metre and t is in second.

(i) Is the particle of medium moving in positive Y or negative Y direction at the origin at time $t = 0$? i.e. what will be produced first- crest or trough ?

(ii) Find the displacement, velocity of the particle and the slope of the wave at 480 m away from the origin at time $t = 2$ s.

(iii) Find the speed of wave.

Solution :

(i) At $x = 0$, starting from $t = 0$, if y increases in the negative direction then the trough will be produced and if y increases in the positive direction then the crest will be produced.

Here, at $x = 0$, $y = 5\sin 30\pi t$. Hence, starting from $t = 0$, here y increases in the positive direction. Hence, first a crest will be produced at the origin.

(ii) Displacement at $t = 2$ s for a particle at $x = 480$ m

$y = 5\sin 30\pi \left(2 - \frac{480}{240} \right)$

$= 5\sin 30\pi(0) = 0$ m

Velocity of the particle,

$v = \frac{dy}{dt} = 150\pi\cos 30\pi \left(t - \frac{x}{240} \right)$

$= 150\pi\cos 30\pi \left(2 - \frac{480}{240} \right)$

$= 150\pi$ m/s

Slope of the wave,

$\frac{dy}{dx} = -\frac{5\pi}{8}\cos 30\pi \left(t - \frac{x}{240} \right)$

$= -\frac{5\pi}{8}\cos 30\pi \left(2 - \frac{480}{240} \right)$

$= -\frac{5\pi}{8}$

(iii) Compare the given equation with,

$y = A\sin 2\pi f \left(t - \frac{x}{v} \right)$

$\therefore$ Wave speed $v = 240$ m/s

Here, note that the wave speed and the magnitude of velocity of the particle taking part in the wave propagation are not equal.

8.7 Wave Speed in Medium

8.7.1 Speed of Transverse Wave on Stretched String :

Earlier we have seen that the particles of the string come back to their original position after the disturbance (or wave) has passed through those particles. In order that particles come back to their original positions, restoring force and hence elasticity in medium are essential. Moreover, the inertia of the medium plays a role in deciding the displacement of the oscillatory particles. **Thus, the elasticity and intertia of medium are necessary for the propagation of the mechanical waves.** From these two properties of medium, the speed of wave in a meduim is determined.

It is found that the speed of transverse wave in a medium like a string kept under the tension, depends on (i) linear mass density (μ) and (ii) tension T in the string.

Here, we will obtain the speed of wave on a string using dimensioned analysis.

Linear density of a string means mass per unit length (μ) of the string.

Dimensional formula of

$$\mu = \frac{[\text{Total mass of string}]}{[\text{Total length of string}]} = \frac{M^1}{L^1}$$

$$= M^1L^{-1}T^0$$

Dimension of Tension $T = M^1L^1T^{-2}$

Suppose, wave speed

$$v = k\ \mu^a\ T^b \qquad (8.7.1)$$

Here, k = dimensionless constant and $[a, b] \in R$.

Substituting dimensions on both the sides,

$$M^0L^1T^{-1} = [M^1L^{-1}T^0]^a\ [M^1L^1T^{-2}]^b$$

$$= M^{a+b}\ L^{-a+b}\ T^{-2b}$$

Comparing dimensions of both the sides, $a + b = 0$, $-a + b = 1$ and $-2b = -1$

$$\therefore\ a = -\frac{1}{2} \text{ and } b = \frac{1}{2}$$

Substituting value of a and b in equations (8.7.1)

$$v = k\ \mu^{-\frac{1}{2}}T^{\frac{1}{2}}$$

From the experimental and other studies, $k = 1$

$$\therefore\ v = \sqrt{\frac{T}{\mu}} \qquad (8.7.2)$$

Above equation shows that wave speed is independent of frequency of a wave and amplitude of a wave.

Illustration 6 : A long wire PQR is made by joining two wires PQ and QR of equal radii. The wire PQ has length 4.8 m and mass. 0.06 kg. The wire QR has length 2.56 m and mass 0.2 kg. The wire PQR is under the tension of 80 N. Find the time taken by a wave produced at the end P to reach the other end R.

Solution :

Mass per unit length for the wire PQ,

$$\mu_1 = \frac{0.06}{4.8} = \frac{1}{80}\ \frac{kg}{m}$$

Mass per unit length for the wire QR,

$$\mu_2 = \frac{0.2}{2.56} = \frac{10}{128}\ \frac{kg}{m}$$

$\therefore$ Speed of wave in the wire PQ,

$$v_1 = \sqrt{\frac{T}{\mu_1}} = \sqrt{\frac{80}{\frac{1}{80}}} = 80 \text{ m/s}$$

$\therefore$ Speed of wave in the wire QR,

$$v_2 = \sqrt{\frac{T}{\mu_2}} = \sqrt{\frac{80}{\frac{1}{128}}} = 32 \text{ m/s}$$

$\therefore$ Time taken by the wave to reach R from P, $t = t_1 + t_2$

$$= \frac{PQ}{v_1} + \frac{QR}{v_2}$$

$$= \frac{4.8}{80} + \frac{2.56}{32}$$

$$= 0.14 \text{ s}$$

Illustration 7 : A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope ?

Solution :

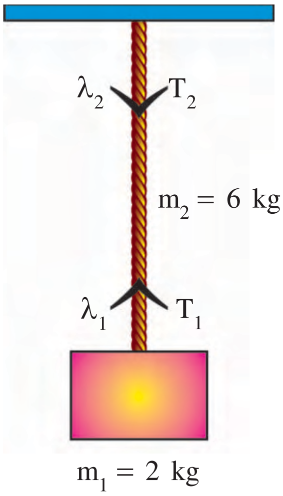


Figure 8.8

As the rope is heavy, its tension will be different at different points.

mass of rope m_2 = 6 kg

mass a block m_1 = 2 kg

Tension at the lower end of rope,

$T_1 = m_1 g = 2g$

Tension of the upper end of rope,

$T_2 = (m_1 + m_2)g$

$= (6 + 2)g = 8g$

wave speed in a string $v = \sqrt{\frac{T}{\mu}}$

$\therefore f\lambda = \sqrt{\frac{T}{\mu}}$ $(\because v = f\lambda)$

The frequency of the wave pulse will be the same everywhere on the rope and μ is also the same throughout the rope as it is uniform. Therefore,

$\lambda \propto \sqrt{T}$

Wavelength of wave at lower end of rope,

$\lambda_1 \propto \sqrt{T_1}$

Wavelength of wave at upper end of rope,

$\lambda_2 \propto \sqrt{T_2}$

$$\therefore \frac{\lambda_2}{\lambda_1} = \sqrt{\frac{T_2}{T_1}}$$

and $\lambda_2 = \lambda_1 \sqrt{\frac{T_2}{T_1}}$

$= (0.06)\sqrt{\frac{8g}{2g}}$

$= 0.12$ m

Illustration 8 : The speed of transverse wave going on a wire having length 50 cm and mass 5.0 g is 80 m/s. The area of cross-section of the wire is 1.0 mm^2 and its Young's modulus is 16×10^{11} N/m^2. Find the extension of the wire over its natural length.

Solution :

Length of the wire L = 50 cm = 50×10^{-2} m

mass of wire m = 5g = 5×10^{-3} kg

cross sectional area of wire

A = $1mm^2 = 1 \times 10^{-6}$ m^2

Young's modulus of wire Y = 16×10^{11} N/m^2

wave speed in a wire v = 80 m/s.

mass per unit length of wire,

$$\mu = \frac{m}{L} = \frac{5 \times 10^{-3}}{50 \times 10^{-2}} = 1 \times 10^{-2} \text{ kg/m}$$

The wave speed in wire $v = \sqrt{\frac{T}{\mu}}$

$\therefore$ Tension in wire = T = F = μv^2

$= (1 \times 10^{-2})(80)^2$

$= 64$ N

Now, Young's modulus $Y = \frac{F/A}{\Delta L/L}$

$\therefore$ Extension in the length of wire,

$$\Delta L = \frac{FL}{AY}$$

$$= \frac{(64)(50 \times 10^{-2})}{(1 \times 10^{-6})(16 \times 10^{11})}$$

$= 0.02$ mm

8.7.2 Speed of sound waves (longitudinal wave) in a medium :

It is found that the speed of longitudinal waves like sound waves in a medium depends on (*i*) the elastic constant E and (*ii*) density ρ of the medium.

Using these facts, we can obtain the speed of the longitudinal waves using dimensional analysis as follows.

Wave speed $v = kE^a\rho^b$

Here, k is dimensionless constant and $[a, b] \in R$.

Now, $[E] = M^1L^{-1}T^{-2}$, $[\rho] = M^1L^{-3}T^0$

Writing dimensional formula on both the sides,

$$M^0L^1T^{-1} = [M^1L^{-1}T^{-2}]^a\ [M^1L^{-3}T^0]^b$$
$$= M^{a+b}\ L^{-a-3b}\ T^{-2a}$$

Comparing dimensions on both the sides,

$a + b = 0,\ -a - 3b = 1$ and $-2a = -1$

$\therefore\ a = \frac{1}{2}$ and $b = -\frac{1}{2}$

$$\therefore\ v = kE^{\frac{1}{2}}\rho^{-\frac{1}{2}}$$

From the experimental and other studies $k = 1$,

$$\therefore\ v = \sqrt{\frac{E}{\rho}} \qquad (8.7.3)$$

The propagation of longitudinal waves in fluid is in the form of condensations and rarefactions. In such a situation due to the variation in pressure of different regions of the medium bulk modulus (B) is taken as the elastic constant.

$$\therefore\ v = \sqrt{\frac{B}{\rho}} \qquad (8.7.4)$$

During the propagation of longitudinal waves in a linear medium like a rod, the longitudinal strain is produced. Hence in such a situation Young's modulus is taken as the elastic constant.

$$\therefore\ v = \sqrt{\frac{Y}{\rho}} \qquad (8.7.5)$$

Table 8.1 gives the speed of sound in various media.

Table 8.1 Speed of sound in some media (Only For Information)

Medium	Speed (m/s)
Gases	
Air (0°C)	331
Air (20°C)	343
Helium	965
Hydrogen	1284
Liquids	
Water (0°C)	1402
Water (20°C)	1482
Seawater	1522
Solids	
Aluminium	6420
Copper	3560
Steel	5941
Rubber	54

It is clear from the Table (8.1) that although the densities of liquids and solids are much greater than those of the gases, the speed of sound in them is higher. It is because liquids and solids are less compressible than gases. i.e have much greater bulk modulus.

Newton's Formula :

Newton assumed that the process of propagation of sound in gas (or air) is isothermal. Hence, the isothermal bulk modulus is to be used in the equation (8.7.4).

For an isothermal process PV = constant

(Taking T = constant, PV = μRT = constant)

Differentiating with respect to V,

$$P\frac{dV}{dV} + V\frac{dP}{dV} = 0$$

$$\therefore\ P = -V\frac{dP}{dV} = -\frac{dP}{dV/V} = \text{Bulk modulus B}$$

Thus, isothermal bulk modulus B = Pressure P.

$$(\because\ B = -\frac{dP}{dV/V})$$

$$\therefore\ \text{Wave speed } v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{P}{\rho}} \qquad (8.7.6)$$

This formula is called Newton's formula for the speed of sound in air.

Illustration 9 : Obtain the speed of sound in air at STP using Newton's formula. Mass of 1 mole of air = 29.0×10^{-3} kg.

$P = 1.01 \times 10^5\ P_a)$

Solution : Volume of 1 mole of air at STP = 22.4 L = 22.4×10^{-3} m^3

Density of air at STP $\rho = \frac{\text{Mass}}{\text{Volume}}$

$$\therefore\ \rho = \frac{29.0 \times 10^{-3}}{22.4 \times 10^{-3}} = \frac{29.0}{22.4}$$

$\therefore$ Speed of sound in air at STP,

$$v = \sqrt{\frac{P}{\rho}}$$

$$= \sqrt{\frac{1.01 \times 10^5 \times 22.4}{29.0}} = 279.3 \text{ m/s}$$

Laplace's Correction :

The speed of sound according to Newton's formula is 279.3 m/s while its experimental value is 332 m/s at STP. This suggests that there is some defect in the formula (8.7.6)

Laplace suggested that the temperature of the region where condensation is formed increases and that of the region of rarefaction decreases. Hence, the process of propagation of sound in a medium cannot be considered isothermal.

The process of formation of condensation and rarefaction in the medium is so quick that the heat produced during the condensation, is absorbed at the same place during rarefaction before being dissipated outside. Relatively small thermal conductivity of gas also helps in not allowing the heat to be dissipated outside. Thus, **the process of sound propagation in the gas is adiabatic and not isothermal.** Hence, adibatic bulk modulus of the gas should be used in place of isothermal bulk modulus.

For an adiabatic process of an ideal gas,

PV^{γ} = constant

Where γ is the ratio of two specific heats C_P and C_V.

Differentiating the equation with respect to V.

$$P \cdot \gamma V^{\gamma-1} + V^{\gamma}\frac{dP}{dV} = 0$$

$$\therefore \gamma P + V\frac{dP}{dV} = 0$$

$$\therefore \frac{-dP}{dV/V} = \gamma P$$

$$\therefore B = \gamma P$$

Thus, for an adiabatic process bulk modulus $B = \gamma P$.

Using this value of B in equation (8.7.4)

$$\text{wavespeed } v = \sqrt{\frac{\gamma P}{\rho}} \qquad (8.7.7)$$

For air γ is 1.41. Speed of sound comes out 331.6 m/s at STP on taking this value of v. This agrees very well with the experimental value. To determine speed of wave in ideal gas Laplace equation (8.7.7) should use instead of Newton's formula.

Various factors affecting speed of sound waves : The equation of state for 1 mole of ideal gas is.

$PV = RT \quad (\mu = 1 \text{ mol})$

$$\therefore P = \frac{RT}{V}$$

Substituting value of P in $v = \sqrt{\frac{\gamma P}{\rho}}$

$$\therefore v = \sqrt{\frac{\gamma RT}{V\rho}}$$

But, ρV = mass of one mole gas = molecular mass M of gas

$$\therefore \text{Speed } v = \sqrt{\frac{\gamma RT}{M}} \qquad (8.7.9)$$

From this expression it is clear that the speed of sound in a gas is directly proportional to the square root of its absolute temperature (T).

i.e. $v \propto \sqrt{T}$

If pressure (P) of the gas is changed keeping its temperature constant, $\frac{P}{\rho}$ remains constant as the density ρ of the gas directly varies as the pressure P. Therefore, the **speed of sound in a gas does not depend on the pressure of the gas** at constant temperature and constant humidity.

Density of water vapour is less than the density of dry air at same pressure. Hence, the **speed of sound increases with increasing humidity** as per $v = \sqrt{\frac{\gamma P}{\rho}}$.

Illustration 10 : Show that the velocity of sound in a gas at temperature t is given by,

$$v_t = v_0\left(1 + \frac{t}{546}\right)$$

Where, v_0 is speed of sound in air at 0° C ($t << 273$)

Solution : The speed of the wave in gas is

$$v = \sqrt{\frac{\gamma RT}{M}}$$

i. e. $v \propto \sqrt{T}$

If, v_t = speed of sound in gas at t° C

v_0 = speed of sound in gas at 0° C

$$\therefore \frac{v_t}{v_0} = \sqrt{\frac{273+t}{273}}$$

$(\because \ T(K) = t(^oC) + 273)$

$$\therefore v_t = v_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}}$$

Using binomial expansion and neglecting higher order terms,

$$v_t = v_0\left(1+\frac{1}{2}\times\frac{t}{273}\right)$$

$$v_t = v_0\left(1+\frac{t}{546}\right)$$

[**Note :** If the speed of sound in air at 0° C is 332 m/s, than speed at 1° C will be,

$$v_t = v_0\left(1+\frac{t}{546}\right)$$

$$= 332\left(1+\frac{1}{546}\right) = 332.61 \text{ m/s}$$

Thus, the velocity of sound in air increases by 332.61 − 332 = 0.61 m/s for every 1° C rise in temperature.]

Illustration 11 : If the velocity of sound in air at 27° C and 76 cm of mercury is 345 m/s. Find the velocity at 127° C and 75 cm of mercury.

Solution : Remember that there is no effect of change of pressure on the velocity of sound.

If v_1 and v_2 be the velocities of sound at 27° C and 127° C, then we have

$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} = \sqrt{\frac{273+127}{273+27}} = \sqrt{\frac{4}{3}}$$

∴ Speed of sound at 127° C,

$$v_2 = v_1 \times \sqrt{\frac{4}{3}} = 345 \times \sqrt{\frac{4}{3}} = 398.4 \text{ m/s}$$

Illustration 12 : The speed of sound in dry air at STP is 332 m s^{-1}. Assume air as composed of 4 part of nitrogen and one part of oxygen. Calculate speed of sound in oxygen under similar condition when the density of oxygen and nitrogen at STP are in the ratio of 16:14.

Solution : Density of air = $\frac{\text{Total mass}}{\text{Total volume}}$

$$\rho_a = \frac{\text{(Mass of oxygen)+(Mass of nitrogen)}}{\text{(Volume of oxygen)+(Volume of nitrogen)}}$$

$$\rho_a = \frac{(V\times\rho_0)+(4V\times\rho_N)}{V+4V}$$

$$= \frac{\rho_0+4\rho_N}{5}$$

$$= \frac{\rho_0\left(1+4\times\frac{\rho_N}{\rho_0}\right)}{5}$$

$$= \frac{\rho_0\left(1+4\times\frac{14}{16}\right)}{5}$$

$$= 0.9\rho_0$$

Speed of sound $v \propto \frac{1}{\sqrt{\rho}}$ $\left(\because v = \sqrt{\frac{\gamma P}{\rho}}\right)$

∴ Speed of sound in oxygen, $v_0 \propto \frac{1}{\sqrt{\rho_0}}$

Speed of sound in air $v_a = \frac{1}{\sqrt{\rho_a}}$

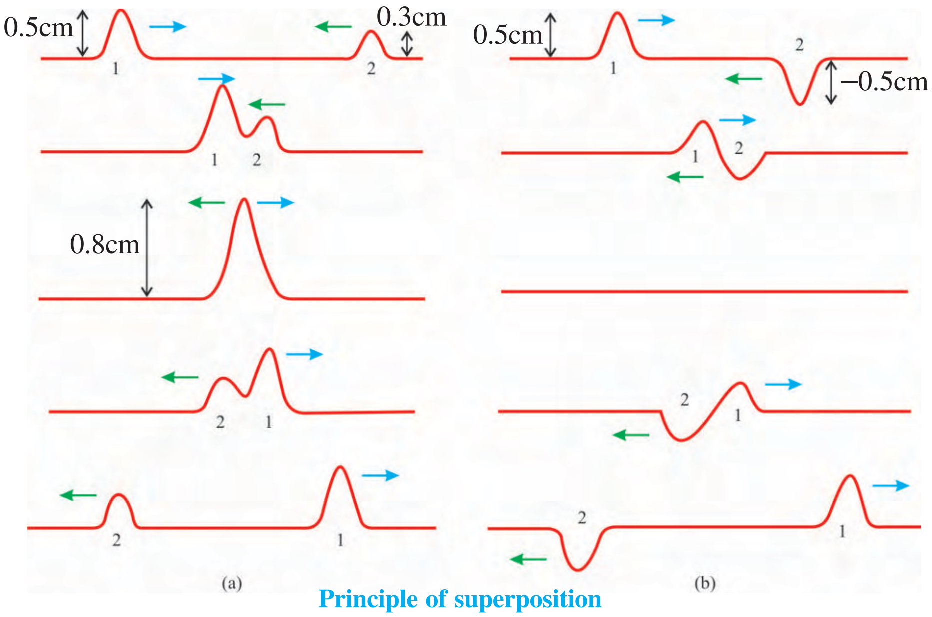
$$\therefore \frac{v_0}{v_a} = \sqrt{\frac{\rho_a}{\rho_0}} = \sqrt{\frac{0.9\rho_0}{\rho_0}} = 0.9487$$

$$\therefore v_0 = v_a \times 0.9487 = 332 \times 0.9487$$

$$= 314.77 \text{ m/s}$$

8.8 Superposition Principle and Reflection of the Wave

So far we have discussed a single wave propagating on a string. Suppose two persons holding the string at the two ends snap their hands once, then two wave pulses will be produced and move towards each other as shown in figure 8.9(*a*). The pulses travel at same speed because the medium is same.



Principle of superposition

Figure 8.9

Suppose the maximum displacement of particle in first wave is 0.5 cm and in second wave is 0.2 cm. As the two wave approaching each other, at any instant both the waves will overlap in some region of the string. Then after they move with their original shape and in their original direction. The net maximum displacement of the particle of string in overlaped region would be 0.5 cm + 0.3 cm = 0.8 cm.

Suppose the two persons snap the end of the string such that wave pulse generates at both the end of the string as shown in Figure 8.9(*b*).

In the first wave pulse the maximum displacement of particle is 0.5 cm in upward and in the other wave pulse the maximum displacement of particle is 0.5 cm in downward direction.

When the wave pulses approach each other, at some instant they overlay on the string and displacement of all the particles will be 0.5cm + (–0.5 cm) = 0. However, the velocities of the particles will not be zero. In this situation, string becomes straight everywhere than both the wave pulses will emerge and move in their original direction.

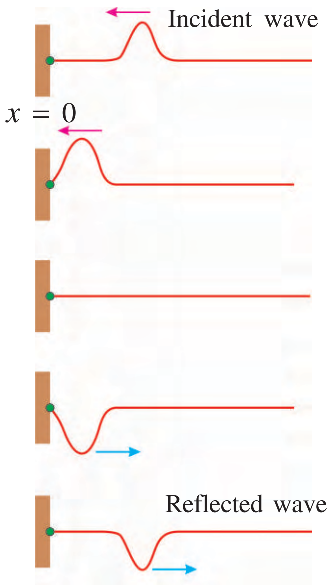
From the above observation principle of superposition can be given as follows.

"When a particle of medium comes under the influence of two or more waves simultaneously, its net displacement is the vector sum of displacement that would occur under the influence of the individual waves."

Reflection of Waves :

(*a*) Reflection of waves from a rigid support :

Suppose a wave propagating in the direction of decreasing value of x, represented by the equation $y = A \sin (\omega t + kx)$ reaches a point $x = 0$, when the wave arrives at the rigid end it exerts a force on the support (wall). By Newton's third law, the support exerts an equal but opposite reaction force on the string. This reaction force generates a wave at the support which travels back along the string in the direction opposite that of incident wave. This wave is known as **reflected wave.**



Reflection of wave from rigid support

Figure 8.10

Oscillation of the particle at point $x = 0$, due to wave $y = A \sin(\omega t + kx)$, can be represented as, $y_i = A \sin\omega t$ (8.8.1)

But, the support at $x = 0$ is fixed, the displacement at $x = 0$ must always be zero. According to principle of superposition, the displacement at $x = 0$ due to reflected wave. It can be give as,

$y_r = -A \sin\omega t$ (8.8.2)

Equation (8.8.2) can be represented as follows :

$y_r = A\sin(\omega t + \pi)$ (8.8.3)

This shows that as the wave reflected from a fixed support, its phase is increased by π. Thus, the 'shape' of the waveform is inverted on reflection. i.e. Crest becomes a trough and trough becomes a crest.

The reflected wave is travelling in the direction of increasing value of x. So the equation can be written as,

$y_r = A \sin(\omega t + \pi - kx)$

$\therefore\ y_r = -A \sin(\omega t - kx)$ (8.8.4)

If the incident wave is travelling in the direction of increasing value of x, then

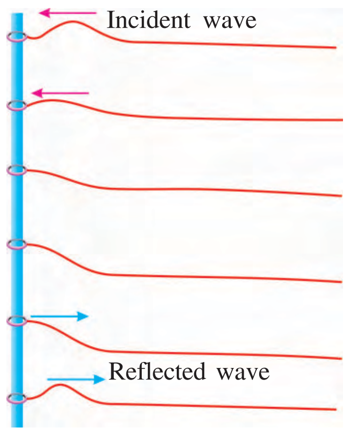
$y_i = A \sin(\omega t - kx)$ (8.8.5)

And equation of reflected wave can be given as,

$y_r = -A \sin(\omega t + kx)$ (8.8.6)

(*b*) Reflection of waves from a free end :

As shown in Figure 8.11 suppose one end of a string is tied to a very light ring which can slide or move without any friction on a vertical rod. Such an end of the ring is said to be free end and here, we will understand the reflection of waves from such a free end.



Reflection of a wave from a free end

Figure 8.11

Suppose the crest like shape of the wave produced from the other end of the string reaches the ring. The ring is then pushed upwards as it is not fixed. Hence the string tied to the ring is also pulled up. As a result of this, now, a reflected wave pulse is generated from this end of the string. Phase of this reflected waves is equal to the phase of the incident wave. So, in this situation the shape is not inverted and a crest is reflected as a crest and a trough as a trough only. Moreover, during such a reflection both the waves are simultaneously present on the ring in same phase and hence the displacement of the ring on the rod is twice the amplitude of the incident wave.

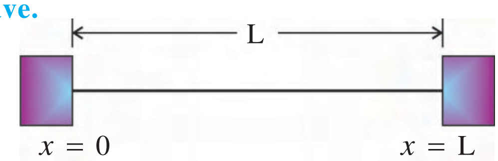
From this discussion it is clear that if the equation of the incident wave is $y_i = A \sin(\omega t + kx)$ then the equation of its reflected wave from a free end will be

$y_r = A \sin(\omega t - kx)$ (8.8.7)

Thus, a travelling wave at a rigid boundary or a closed end, is reflected with a phase reversal of π but the reflection at an open boundary takes place without any phase change.

8.9 Stationary Waves

When two waves having the same amplitude and frequency (i.e. wavelength) and travelling in mutually opposite directions are superposed, the resultant wave formed loses the property of propagation and a stationary pattern is created in the medium. Such a wave is called a **stationary wave.**



A string fixed at both the ends with rigid supports

Figure 8.12

To understand stationary waves, consider a string of length L, kept under a suitable tension, fixed at its two ends. The harmonic waves produced in this string will be reflected from rigid supports repeatedly so that each element of string is under the influence of "incident" and "reflected" waves.

Let the wave propagating in the direction of increasing x be

$$y_1 = A \sin(\omega t - kx) \quad (8.9.1)$$

The reflected wave propagating in the direction of decreasing x will then be,

$$y_2 = -A \sin(\omega t + kx) \quad (8.9.2)$$

According to principle of superposition, the displacement of a particle of a string is given by,

$$y = y_1 + y_2$$
$$= A \sin(\omega t - kx) - A \sin(\omega t + kx)$$

Now,

$$\therefore y = -2A\cos\omega t \sin kx \text{ (see foot note)}$$
$$= -2A \sin kx \cos\omega t \quad (8.9.3)$$

The functional form of this wave is not of type $f(\omega t \pm kx)$ which means that it is not a travelling or progressive wave. Equation (8.9.3) is an equation of a stationary wave. **Energy does not propagate in this type of a wave and hence it is named as a stationary wave.**

The term 'cos ωt' of the equation (8.9.3) shows that each particle of the string is executing a simple harmonic motion and their amplitudes depends upon position x according to $2A\sin kx$. Here, amplitudes of all the particles are not same.

The location of particles for which $\sin kx = 0$, have zero amplitude and these points remains stationary. These points are called **'Nodes'**.

The positions in a stationary wave where the amplitude always remains zero are called the 'Nodes'.

Now $\sin kx = 0$

$$\therefore kx = n\pi \qquad \text{where } n = 1, 2, 3 \ldots\ldots$$

$$\therefore x = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda}$$

$$\therefore x = \frac{n\lambda}{2} \quad (8.9.4)$$

This shows that the nodes are located at a distance $x = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots\ldots \frac{n\lambda}{2}$ from the end $x = 0$. **The distance between the successive node is $\frac{\lambda}{2}$.**

Foot note : $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Maximum amplitudes occur at points for which $\sin kx = \pm 1$. These points are called "Antinodes".

The positions in a stationary wave where the amplitude always remains maximum are called the 'Antinodes'.

$\sin kx = \pm 1$

$$\therefore kx = (2n-1)\frac{\pi}{2} \quad \text{where, } n = 1, 2, \ldots\ldots$$

$$\therefore x = \frac{(2n-1)\pi}{2k}$$

$$= (2n-1)\frac{\lambda}{4} \quad (8.9.5)$$

Thus, the antinodes are located at $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots\ldots$ from the end $x = 0$. **Distance between successive antinode is also $\frac{\lambda}{2}$. Distance between a node and an adjacent antinode is $\frac{\lambda}{4}$.**

In Figure 8.13 the antinodes are shown as A and the nodes are shown as N.

The displacement of string at the end $x = 0$ and at the end $x = L$ is always zero because the string is fixed to a rigid suppot at $x = L$.

$$\therefore \sin kL = 0$$

$$\therefore kL = n\pi \qquad \text{where } n = 1, 2, 3, \ldots\ldots$$

$$\therefore \frac{2\pi}{\lambda}L = n\pi$$

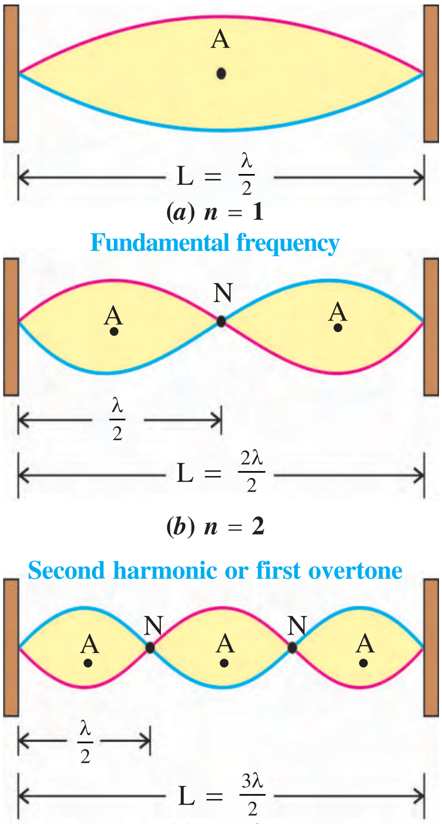
$$\therefore \lambda_n = \frac{2L}{n} \quad (8.9.6)$$

This equation shows that for a string at given length L, stationary wave can be formed only with waves having specific discrete values for their wave length like $2L, L, \frac{2L}{3}, \frac{L}{2}, \ldots\ldots$ appropriate to different values of n. Thus, waves with arbitrary wavelength cannot form stationary waves on a string of a given length.

The frequency of the standing waves produced on a string will have corrosponding to its restricted wavelength. It is given by,

$$f_n = \frac{v}{\lambda_n}$$

$$\therefore f_n = \frac{nv}{2L} \text{ (from equation 8.9.6)} \quad (8.9.7)$$



Fundamental frequency

Second harmonic or first overtone

Third harmonic or second overtone

Stationary waves on a string

Figure 8.13

or $f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$ (8.9.8)

Where, v = speed of wave on string = $\sqrt{\frac{T}{\mu}}$

Substituting $n = 1$ in equation (8.9.7)

$$f_1 = \frac{v}{2L}$$

Here, f_1 is called **fandamental frequency** or **first harmonic.**

Taking $n = 2$,

$$f_2 = \frac{2v}{2L} = 2f_1$$

f_2 is called **second harmonic** or **first overtone.**

Taking $n = 3$,

$$f_3 = \frac{3v}{2L} = 3f_1$$

f_3 is called **third harmonic** or **second overtone**

In this way taking successive integral values of n, all possible oscillation of the string are obtained and corresponding frequencies of the fourth, fifth etc. harmonics are obtained.

Figure 8.13 shows oscillation of string with first, second and third harmonics. Form figure it is clear that number of loops produced on the string is same as value of n.

These oscillations with discrete frequencies in various harmonics are called the **'Normal Modes of Oscillation of a system.**

Frequencies appropriate to the different normal modes of vibrations can be obtained from the following equation.

$$f_n = \frac{nv}{2L} = nf_1 \quad \text{where } n = 1, 2, 3....$$

Here, f_n is the frequency of wave produced on a string. It is also called nth harmonic or $(n - 1)$th overtone. The integer n indicates the number of loops on the string.

Illustration 13. : The stationary waves produced in a 60 cm long string tied at both the ends with rigid support are represented by $y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t)$. Here, x and y are in cm and t is second. Find out,

(1) position of nodes,

(2) positions of anti-nodes,

(3) maximum displacement of the particle at $x = 5$ cm

(4) the equation of the component waves.

Solution : Comparing

$$y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t) \text{ with}$$

$$y = 2A\sin(kx)\cos(\omega t),$$

$$A = 2 \text{ cm}, k = \frac{\pi}{15}\frac{\text{rad}}{\text{cm}} \text{ and } \omega = 96\pi \text{ rad/s}$$

$$\text{But, } k = \frac{2\pi}{\lambda}$$

$$\therefore \frac{2\pi}{\lambda} = \frac{\pi}{15} \Rightarrow \lambda = 30 \text{ cm}$$

(1) Positions of nodes

$$= \frac{n\lambda}{2}, \quad \text{where } n = 1, 2,...$$

$$= 15 \text{ cm}, 30 \text{ cm}, 45 \text{ cm}$$

(The particles at 0 cm and 60 cm are tied to the rigid supports and hence they are not considered here.)

(2) Positions of antinodes,

$= (2n - 1)\frac{\lambda}{4}$, where $n = 1, 2, 3, ...$

$= 7.5$ cm, 22.5 cm, 37.5 cm, 52.5 cm

(3) Maximum displacement of the particle at a distance

$x = 2A\sin kx$

$= 4\sin\left(\frac{\pi x}{15}\right)$

$= 4\sin\left(\frac{\pi}{3}\right)$ ($\because x = 5$ cm)

$= 4\frac{\sqrt{3}}{2}$

$= 2\sqrt{3}$ cm

(4) $y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t)$

$= 2\sin\left(\frac{\pi x}{15} + 96\pi t\right) +$

$2\sin\left(\frac{\pi x}{15} - 96\pi t\right)$

$\therefore$ Component of waves are

$y_1 = 2\sin\left(\frac{\pi x}{15} + 96\pi t\right)$cm and,

$y_2 = 2\sin\left(\frac{\pi x}{15} - 96\pi t\right)$cm

Illustration 14 : The equation of a progressive, harmonic waves travelling in a medium is given by an equation $y_i = A\cos(ax + bt)$; where A, a and b are positive constants. This wave is reflected from a rigid support kept at $x = 0$. The intensity of the reflected wave is 0.64 times that of the incident wave.

(a) What are wavelength and frequency of the incident wave ?

(b) Write the equation of the reflected wave.

(c) Express the resultant wave in the form of progressive and stationary waves.

Solution :

(a) Incident wave is $y_i = A\cos(ax + bt)$

Comparing this equation with the wave-equation $y = A\cos(kx + \omega t)$,

$\therefore$ wave-vector $k = a$

$\therefore \frac{2\pi}{\lambda} = a$

$\therefore \lambda = \frac{2\pi}{a}$

Angular frequency $\omega = 2\pi f = b$

$\therefore f = \frac{b}{2\pi}$

(b) Intensity $I \propto A^2$, where A = amplitude.

Suppose amplitudes of the incident and the reflected waves are A_1 and A_2 respectively and I_1 and I_2 are their intensities respectively.

$$\therefore \frac{I_2}{I_1} = \frac{(A_2)^2}{(A_1)^2}$$

$$\therefore \frac{A_2}{A_1} = \left(\frac{I_2}{I_1}\right)^{\frac{1}{2}} = (0.64)^{\frac{1}{2}}$$

$\therefore A_2 = 0.8$ A ($\because A_1$ = Amplitude of the incident wave = A)

$\therefore$ Amplitude of the reflected wave $A_2 = 0.8$ A

Equation of the reflected wave

$y_r = -A_2\cos(bt - ax)$

$\therefore y_r = -0.8\,A\cos(bt - ax)$

(c) Resultant wave $y = y_i + y_r$

$= A\cos(bt + ax) - 0.8\,A\cos(bt - ax)$

$= 0.8\,A[\cos(bt + ax) - \cos(bt - ax)]$

$+ 0.2\,A\cos(bt + ax)$

$= -1.6\,A\sin(ax)\,.\,\sin(bt)$

$+ 0.2\,A\cos(bt + ax)$,

where, stationary wave,

$y_s = -1.6\,A\sin(ax)\,.\,\sin(bt)$ and

progressive wave $y_p = 0.2\,A\cos(bt + ax)$

Illustration 15 : A block is attached to the free end of a sonometer wire. The wire has fundamental frequency f_1 Hz in this situation. Now the block is immersed in water and it is found that the wire has a fundamental frequency f_2 Hz. When the block is immersed in some liquid, the fundamental frequency of the wire is f_3 Hz. Find the specific gravity of the material of the block and that of the liquid.

Solution : The force of buoyancy is different when the block is in air, in water and in liquid. So the effective weight is different in these cases. Hence, tension in the wire is also different and as a result the frequency is also different for the wire of same length and same material.

Suppose, the weight of block in air is W_1, in water W_2 and in liquid W_3.

Fundamental frequency, $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$

Here, L and μ being constant,

$f \propto \sqrt{T}$

$\therefore$ $T = kf^2$ where, k = constant of proportionality.

But, tension T = Weight W

$\therefore$ $W = kf^2$

$\therefore$ $W_1 = kf_1^2$; $W_2 = kf_2^2$; $W_3 = kf_3^2$

According to Archimedes' principle,

Specific gravity of block

$$= \frac{\text{Weight of block in air}}{\text{Loss of weight of block in water}}$$

$$= \frac{W_1}{W_1 - W_2} = \frac{f_1^2}{f_1^2 - f_2^2}$$

Specific gravity of liquid

$$= \frac{\text{Loss of weight of block in liquid}}{\text{Loss of weight of block in water}}$$

$$= \frac{W_1 - W_3}{W_1 - W_2} = \frac{kf_1^2 - kf_3^2}{kf_1^2 - kf_2^2}$$

$$= \frac{f_1^2 - f_3^2}{f_1^2 - f_2^2}$$

8.10 Stationary Wave in Pipes

As stationary waves are formed on a string due to superposition of incident and reflected transverse waves of definite frequencies, stationary waves are also formed due to reflection of longitudinal waves of definite frequencies in the air column, from the end of a pipe. The flute trumpet, clarionet etc. are the musical instrument that are organ pipes in which stationary longitudinal waves are formed. Such pipes are of two types : (1) an open pipe in which both ends are open e.g. flute. (2) a closed pipe in which one end is closed, e.g. clarionet.

Just as in case of string, a node obtained at the fixed end, for a closed pipe a **node** is always formed at the closed end because longitudinal waves are reflected from closed end. If the pipe is narrow compared to the wavelength of wave, an **antinode** is formed at the open end (slightly outside). The situation is slightly complicated for the reflection of longitudinal waves at the open end of the pipe.

Stationary Waves in a Closed Pipe :

For stationary waves to be formed in a closed pipe the wavelength (λ) of the wave should be such that a node is formed at the closed end of the pipe and an antinode at its open end. In stationary waves the distances between nodes and antinodes are $\frac{\lambda}{4}$, $\frac{3\lambda}{4}$, $\frac{5\lambda}{4}$, ...$(2n-1)$ $\frac{\lambda}{4}$, where n = 1, 2, 3,....

Similarly, in general stationary waves is produced in a pipe of length L for wavelength λ only when,

$$L = (2n - 1)\frac{\lambda}{4} \quad (8.10.1)$$

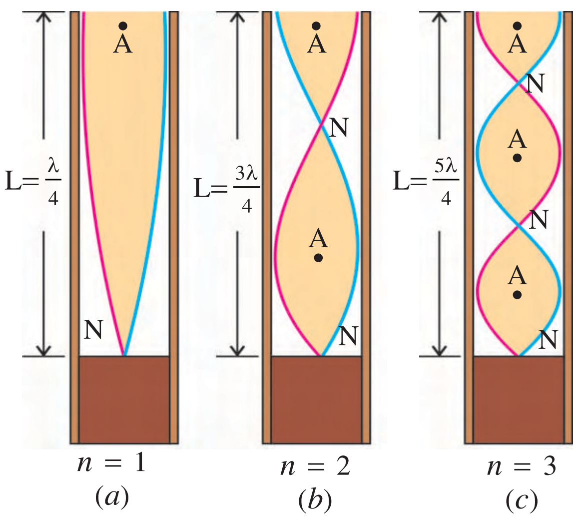
where n = 1, 2, 3, ...

In a closed pipe the value of possible wavelengths required for stationary waves are given by,

$$\lambda_n = \frac{4L}{(2n-1)} \quad (8.10.2)$$

The frequency of stationary waves in pipe will be,

$$f_n = \frac{v}{\lambda_n}$$



n = 1 Fundamental frequency (first harmonic)

n = 2 Third harmonic (first overtone)

n = 3 Fifth harmonic (second overtone)

Stationary waves in a closed pipe

Figure 8.14

$$\therefore f_n = \frac{v}{4L}(2n - 1) \qquad (8.10.3)$$

where, v is a speed of wave.

(i) Taking $n = 1$,

$$f_1 = \frac{v}{4L}$$

f_1 is known as **fundamental frequency** or the **first harmonic.**

(ii) Taking $n = 2$,

$$f_2 = \frac{3v}{4L} = 3f_1 \qquad (\because f_1 = \frac{v}{4L})$$

f_2 is known as **third harmonic** or **first overtone.**

(iii) Similarly for $n = 3$,

$$f_3 = \frac{v}{4L}(2(3) - 1) = \frac{5v}{4L} = 5f_1$$

f_3 is known as fifth harmonic or second overtone.

In general, the frequency of n^{th} mode of normal oscillation in closed pipe is given by,

$$f_n = \frac{v}{4L}(2n - 1)$$
$$= (2n - 1)f_1 \qquad (8.10.4)$$

where $n = 1, 2, 3,....$

Here, f_n represents $(2n - 1)$ harmonic or $(n - 1)^{th}$ overtone.

Thus, in the closed pipe all the harmonics are not possible. The harmonics are possible only for odd multiples of fundamental frequency (f_1, $3f_1$, $5f_1$, ...).

[In this reference, equation (8.10.3) can be written as,

$$f_n = nf_1 = \frac{nv}{4L} \quad \text{where } n = 1, 3, 5....$$

where, f_n represent nth harmonic or $\left(\frac{n-1}{2}\right)$th overtone]

The frequencies for which stationary waves are formed are called natural or characteristic frequencies of the given pipe.

Stationary waves in an open pipe :

In an open pipe, antinodes are formed at both the ends. We know that in stationary waves distances of antinodes are $\frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$ ..... $\frac{n\lambda}{2}$.

Where $n = 1, 2, 3,.......$

Therefore, in an open pipe of length L, the stationary waves can be produced only of those wavelength λ for which,

$$L = \frac{n\lambda}{2}$$

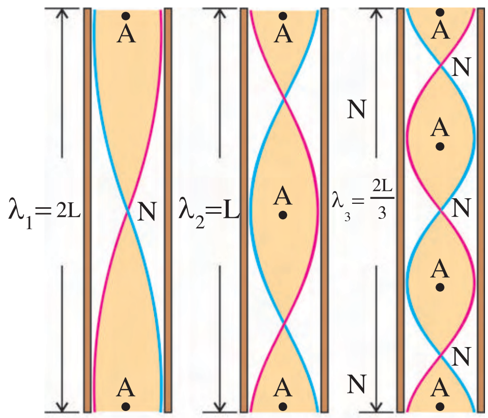
So, possible wavelenghts in pipe will be,

$$\lambda_n = \frac{2L}{n} \qquad (8.10.5)$$

The frequency of a stationary waves in open pipe will be,

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L} \qquad (8.10.6)$$

where, v is speed of a wave.



$n = 1$ (*a*) Fundamental frequency or first harmonic

$n = 2$ (*b*) Second harmonic (first overtone)

$n = 3$ (*c*) Third harmonic (second overtone)

Stationary waves in open pipes

Figure 8.15

(i) Substituting $n = 1$ in equation (8.10.6)

$$f_1 = \frac{v}{2L} \qquad (8.10.7)$$

Here, f_1 is called the **fundamental frequency** or the **first harmonics** (See Figure 8.15*a*) which is double than the fundamental frequency of a closed pipe. ($\because f_1 = \frac{v}{4L}$).

(ii) Taking $n = 2$,

$$f_2 = \frac{2v}{2L} = \frac{v}{L} = 2f_1$$

f_2 is called the **second harmonics** or **first overtone.**

Thus, taking different values of n in equation (8.10.6) third, fourth ..... harmonics can be obtained. In general, for an open pipe the nth harmonic or $(n - 1)$th overtone,

$$f_n = \frac{nv}{2L} = nf_1 \qquad (8.10.8)$$

where, $n = 1, 2, 3....$

Thus, all the harmonics (f_1, $2f_1$, $3f_1$....) are possible for an open pipe.

Thus, in both the types of the pipes there are normal mode of oscillation for the air columns of the pipes.

Illustration 16 : If the second overtone of a closed pipe and third overtone of an open pipe are same, find the ratio of their lengths.

Solution :

For a closed pipe second overtone means fifth harmonics. Put $n = 5$, in following equation,

$$f = \frac{nv}{4L} = \frac{5v}{4L_1}$$

for an open pipe, third overtone means fourth harmonics. Put $n = 4$ in following equation,

$$f = \frac{nv}{2L} = \frac{4v}{2L_2}$$

Now, the frequency is same for both the pipes.

$$\frac{5v}{4L_1} = \frac{4v}{2L_2}$$

$$\therefore \frac{L_1}{L_2} = \frac{5}{8} \text{ OR } L_1 : L_2 = 5 : 8$$

Illustration 17 : Air column of a resonance tube resonates with a tuning fork of frequency 800 Hz when its lengths is 9.75 cm. If the length of the air column is increased to 31.25 cm then also it resonates with the same tuning fork. Find speed of sound in air.

Solution : In the experiment of resonance tube, the arrangement of a closed pipe is obtained by immersing one end of a pipe in the water.

When oscillations are produced in the air column with the help of a tuning fork having frequency equal to the natural frequency of air column it oscillates with large amplitude, and large intense of sound is heard. This is a phenomenon of **resonance.**

Here, $f = 800$ Hz, $L_1 = 9.75$ cm, $L_2 = 31.25$ cm

Resonance tube is a closed pipe. For a closed pipe the natural frequency is given by

$$f = (2n - 1)\frac{v}{4L}$$

Taking $n = 1$ for first resonance,

$$f = \frac{v}{4L_1}$$

$$\therefore L_1 = \frac{v}{4f}$$

Taking $n = 2$ for second resonance,

$$f = (2 \times 2 - 1)\frac{v}{4L_2} = \frac{3v}{4L_2}$$

$$\therefore L_2 = \frac{3v}{4f}$$

$$\therefore L_2 - L_1 = \frac{3v}{4f} - \frac{v}{4f} = \frac{2v}{4f} = \frac{v}{2f}$$

$\therefore$ Speed of sound $v = (L_2 - L_1)(2f)$

$= (31.25 - 9.75)(2 \times 800)$

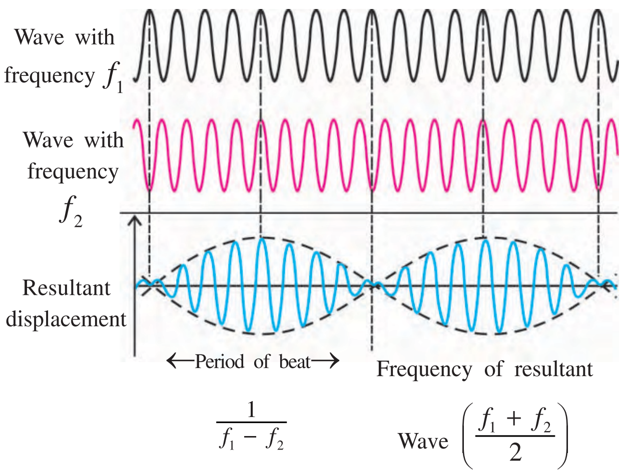
$= 34400$ cm/s

$= 344$ m/s

8.11 Beats

So far we have applied principle of superposition to two waves propagating in opposite direction with equal amplitude and equal frequency. It produces the non-progressive wave like stationary waves.

Let us now consider two waves having equal amplitudes and travelling in a medium in the same direction but having slightly different frequencies. Now we will apply principle of superposition to study the oscillation of a particle of a medium.



Beats

Figure 8.16

Suppose, two harmonic waves superpose at a particular position in the medium are,

y_1 = A sin $\omega_1 t$ = A sin $2\pi f_1 t$ and

y_2 = A sin $\omega_2 t$ = Asin $2\pi f_2 t$

Here, initial phase of both waves is zero. f_1 and f_2 are the frequency of first wave and second wave respectively.

Remember that here we are locally observing the effect of superposition of two waves on any one particle.

Suppose, at time t, the resultant displacement of a given particle is y then according to superposition principle,

$y = y_1 + y_2$

$= A \sin 2\pi f_1 t + A \sin 2\pi f_2 t$

$$\therefore\ y = [2A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t]\ \sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t \qquad (8.11.1)$$

$$y = A'\sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t$$

$$\text{or } y = A'\sin 2\pi\, ft \qquad (8.11.2)$$

Above equation shows that the resultant oscillations of a given particle are the oscillation with a frequency $f = \left(\frac{f_1 + f_2}{2}\right)$. Here, f is the average of the two combining frequencies. The resultant amplitude is,

$$A' = 2A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \qquad (8.11.3)$$

and it changes periodically with time. Here, amplitude is a periodic function of time. Its frequency is $\left(\frac{f_1 - f_2}{2}\right) = f'$

Therefore, the period of oscillation is,

$$T = \frac{1}{f'} = \frac{2}{f_1 - f_2} \qquad (8.11.4)$$

In time period T, the 'cosine' function attains its maximum value and zero twice. Hence, this function becomes $f_1 - f_2$ times maximum in unit time. Therefore, the amplitude of oscillations becomes $f_1 - f_2$ times maximum and $f_1 - f_2$ times zero in unit time.

Foot Note : $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

If these waves are sound waves, then loudness of sound is proportional to the square of the amplitude ($I \propto A^2$), the loudness of sound also becomes $f_1 - f_2$ times maximum and $f_1 - f_2$ times zero in unit time.

Thus, phenomenon of the loudness of sound becoming maximum periodically due to superposition of two sound waves of equal amplitude and slightly different frequencies is called the 'beats'. The number of beats in unit time is $f_1 - f_2$. It is also called frequency of beat.

Note : In case of sound waves, in order to hear the beats clearly, $f_1 - f_2$ should not exceed about 6 to 7.

The phenomenon of beats can be experienced by taking two tuning forks of the same frequency and putting some wax on the prongs of one of the forks. Loading with wax decreases the frequency of a tuning fork a little. (By filling one of the prongs of a fork, its frequency will increase a little) When these two forks are vibrated and kept side by side, the listener can recognise the periodic variation of loudness of resulting sound. Musician tune their different musical instruments with the help of beat phenomenon.

Illustration 18 : When two tuning forks A and B were sounded together, 20 beats were produced in 8 seconds. After loading one of the tuning forks with a little wax, they produce 32 beats in 8 seconds. If the unloaded fork had a frequency of 512 Hz. calculate the frequency of the other.

Solution : Suppose, tuning fork B is loaded with wax. Frequency of tuning fork A,

f_A = 512 Hz

Frequency of tuning fork B, f_B = ?

Before loading wax, number of beats per second,

$$= \frac{20}{8} = 2.5 \text{ Hz}$$

$\therefore$ Frequency of B before loading wax either 512 + 2.5 = 514.5 Hz

or 512 − 2.5 = 509.5 Hz

After loading wax on B,

beats per second $= \frac{32}{8} = 4$ Hz.

Frequency of B after loading is,

either $512 + 4 = 516$ Hz

or $512 - 4 = 508$ Hz

Since, after loading the wax frequency of a tuning fork B is lowered. In above calculation we can see that before loading wax, frequency of B is 509.5 Hz and after loading it is 508 Hz.

Hence, original frequency of B (i.e. before loading) will be 509.5 Hz.

8.12 Doppler Effect

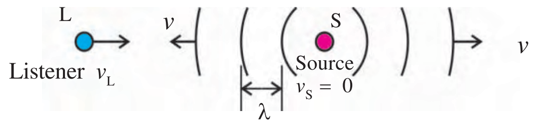
Whenever there is a relative motion between a source of a sound and a listener with respect to medium in which the waves are propagating, the frequency of sound experienced by the listener is different from that which is emitted by the source. This phenomenon is called **Doppler effect.** This effect was discovered by Austrian physicist Johann Christian Doppler (1803–1853).

The frequency of sound of a whistle of the train is found to be more than original frequency and hence its sound appears more shrill (of higher pitch) when the train is approaching you. When it is passing by you, the frequency of sound experienced is same of that of actual sound emitted, and when the train is receding from you, the frequency listens lower than actual and sound appears less shrill than the actual.

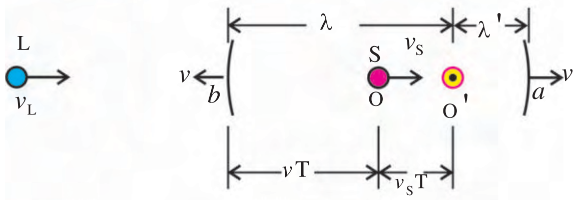
To understand Doppler effect, consider, as shown in Fig. 8.17, a listener moving with velocity v_L and a source of sound moving with a velocity v_S along straight line with respect to stationary air.

As a convention, the velocities in the direction from listener to source are considered as positive and from the source to the listener are considered as negative. The speed of sound is always considered positive. With this convention, we will obtain a general result from which other cases can be obtained easily.

Moving Listener : Suppose, a listener L moving with velocity v_L towards a stationary sources S. (See Figure 8.17a) The source emits a sound wave with frequency f_s and wavelength $\lambda = \frac{v}{f_S}$. Where, v is the speed of sound wave in air.



(*a*)



(*b*)

Doppler effect

Figure 8.17

These waves are travelling towards the listener. Hence, the speed of waves travelling towards the listener, relative to the listener will be $v + v_L$. So the frequency f_L listened by the listener will be,

$$f_L = \frac{v + v_L}{\lambda} \qquad (8.12.1)$$

Moving Source and Moving Listener : Now suppose the source is also moving with velocity v_s in the direction of L to S. (See Figure 8.17 *b*)

Let the source of sound (S) be at O at time $t = 0$ and at O′ at time $t = T$. Where, $T = \frac{1}{f_S}$ is the periodic time of emitted sound.

Now, the distance travelled by the source in time T will be,

$$OO' = v_S T$$

The wave (crest) emitted by the source at $t = 0$ will cover the distance vT in time T. From the figure, Oa = Ob = vT

Now, at time $t = T$, the source is at O′ and it emits successive wave (crest). The wave moving towards the listener will be in the region O′b and the wave moving away from the listener will be in region O′a.

The wavelength of the wave moving towards the listner,

λ = Distance betwen successive wave (crest) in region O'b

$= v_S T + vT$

$$\therefore \lambda = \frac{v_S + v}{f_S} \quad (\because T = \frac{1}{f_S}) \qquad (8.12.3)$$

Substituting value of λ in equation (8.12.1)

$$f_L = \frac{v + v_L}{v + v_S} \cdot f_S \qquad (8.12.3)$$

$$\text{or} \quad \frac{f_L}{v + v_L} = \frac{f_S}{v + v_S} \qquad (8.12.4)$$

From the Figure (8.17) it is clear that the waves in the front of the source (region O'a) are compressed, hence the wavelength is decreasing due to motion of the source, while behind the source (region O'b) waves are stretched out hence its wavelength is increasing. Here, waves are travelling in the same medium (air), then why their wavelengths are changing ? Think over it. The relative displacement of source and wave is responsible for that.

Some Special Cases :

(i) Listener is stationary and source is moving towards the listener, then according to the accepted conventions, taking $v_S = -v_S$ and $v_L = 0$ in equation (8.12.3), the frequency listened by listener,

$$f_L = \frac{v}{v - v_S} f_S$$

i.e. listener will listen the frequency higher than the actual frequency ($f_L > f_S$).

(ii) Listener is stationary and source is moving away from the listener, then $v_L = 0$ and $v_S = +v_S$,

Frequency listened by listener $f_L = \dfrac{v}{v + v_S} f_S$

This shows that $f_L < f_S$. i.e. Listener will hear the lower frequency than actual frequency.

(iii) Both the source and listener are approaching each other, then $v_L = +v_L$ and $v_S = -v_S$ Therefore,

$$f_L = \frac{v + v_L}{v - v_S} f_S$$

In this case, $f_L > f_S$

(iv) Both the source and listener are moving aways from each other, then $v_L = -v_L$ and $v_S = +v_S$

$$\therefore f_L = \frac{v - v_L}{v + v_S} f_S$$

In this case, $f_L < f_S$

In all these cases the medium (air) is considered stationary. If wind is blowing from the source to the listener (in the direction of velocity of sound) with velocity v_w, the velocity of sound will be $v + v_w$ and if the wind is blowing in opposite direction to the motion of sound waves, the velocity of sound will be $v - v_w$.

Moreover, it is assumed that the velocities of the listener and of the source are less than the velocity of sound.

Illustration 19 : A police siren emits a wave with frequency 300 Hz. The speed of sound is 340 m/s. (*a*) Find the wavelength of the waves in the air if the police car is at rest. (*b*) If the police car is moving at 108 km/h, find the wavelength of the waves in front and behind the car.

Solution : (*a*) When the police car is at rest, $f_S = 300$ Hz, $v = 340$ m/s

Wavelength of the waves emitted from the siren.

$$\lambda = \frac{v}{f_S} = \frac{340}{300} = 1.13 \text{ m}$$

(*b*) Speed of a police car $v_S = 108$ km/h $= 30$ m/s

$$\text{Now, } f_L = \frac{v + v_L}{v + v_S} f_S$$

If the listener is in the region of the front of the moving car, then $v_L = 0$, and $v_S = -v_S$

$$\therefore f_{front} = \frac{v}{v - v_S} f_S$$

$$\therefore \frac{v}{\lambda_{\text{front}}} = \frac{v}{v - v_S} f_S$$

$$\therefore \lambda_{\text{front}} = \frac{v - v_S}{f_S} = \frac{340 - 30}{300} = 1.033 \text{ m}$$

For behind the police car,

$v_L = 0$ and $v_S = +v_S$

$$f_{\text{behind}} = \frac{v}{v + v_S} f_S$$

$$\therefore \lambda_{\text{behind}} = \frac{v + v_S}{f_S} = \frac{340 + 30}{300} = 1.233\text{m}$$

Illustration 20 : A SONAR system fixed in a stationary submarine in the sea operates at a frequency 40 kHz. An enemy submarine moves towards the SONAR with a speed of 360 kmh^{-1}. What is the frequency of sound reflected by the submarine ? The speed of sound in water is 1450 m s^{-1}.

Solution : $f_S = 40$ kHz, $v = 1450$ m/s

The frequency of the waves from the SONAR will undergo a change in frequency in two steps.

(i) When the waves are moving towards the enemy's submarine which is in motion, the frequency of waves will change. In this case SONAR is a source (S) and submarine is a listener (L).

Therefore, $v_S = 0$ and

$$v_L = 360 \text{ km/h} = \frac{360 \times 1000}{3600} = 100 \text{ m/s}$$

$$\text{Now, } f_{L_1} = \frac{v + v_L}{v + v_S} \times f_S$$

$$= \frac{1450 + 100}{1450 + 0} \times 40 \times 10^3$$

$$= 42.758 \text{ kHz}$$

(ii) The enemy submarine will reflect waves of frequency 42.758 kHz and will act as a source of waves, while SONAR will act as a listener (L).

$f_S = 42.758$ kHz, $v_L = 0$, $v_S = -100$ m/s

Frequency of reflected wave,

$$f_{L_2} = \frac{v + v_L}{v + v_S} \times f_S$$

$$= \frac{1450 + 0}{1450 - 100} \times 42.758 \times 10^3$$

$$= 45.92 \text{ kHz}$$

Thus the frequency of reflected waves from submarine, moving towards SONAR is 45.92 kHz.

SUMMARY

1. **Waves :** The motion of the disturbance in the medium (or in free space) is called wave pulse or generally a wave.
2. **Amplitude of a wave :** Amplitude of oscillation of particles of the medium is called the amplitude of a wave.
3. **Wavelength and frequency :** The linear distance between any two points or particles having phase difference of 2π rad is called the wavelength (λ) of the wave.
   Frequency of wave is just the frequency of oscillation of particles of the medium.
   Relation between wavelength and frequency :
   $v = f\lambda = \frac{\omega}{k}$, where, v is the speed of wave in the medium.
4. **Mechanical waves :** The waves which require elastic medium for their transmission are called mechanical waves. e.g. sound waves.
5. **Transverse and longitudinal waves :** Waves in which the oscillations are in a direction perpendicular to the direction of wave propagation are called the transverse wave.

Waves in which the oscillations of the particles of medium are along the direction of wave propagation are called longitudinal waves.

6. **Wave Equation :** The equation which describe the displacement for any particle of medium at a required time is called wave equation. Various forms of wave equations are as follows :

(i) $y = A \sin(\omega t - kx)$ (ii) $y = A \sin\left(\frac{t}{T} - \frac{x}{\lambda}\right)$

(iii) $y = A \sin 2\pi f\left(t - \frac{x}{v}\right)$ (iv) $y = A \sin \frac{2\pi}{\lambda}(vt - x)$

The above equations are for the wave travelling in the direction of increasing value of x. If the wave is travelling in the direction of decreasing value of x then put '+' instead of '−' in above equations.

7. The elasticity and inertia of the medium are necessary for the propagation of the mechanical waves.

8. The speed of the transverse waves in a medium like string kept under tension, $v = \sqrt{\frac{T}{\mu}}$

where, T = Tension in the string and μ = mass per unit length of the string = $\frac{m}{L}$

9. Speed of sound waves in elastic medium, $v = \sqrt{\frac{E}{\rho}}$

where, E = Elastic constant of a medium, ρ = Density of the medium.

Speed of longitudinal waves in a fluid, $v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma P}{\rho}}$

where, B = Bulk modulus of a medium $\gamma = \frac{C_P}{C_V} = 1.41$ (for air)

Speed of longitudinal waves in a linear medium like a rod, $v = \sqrt{\frac{y}{\rho}}$

where, y = Young modulus, ρ = Density of a medium

At constant pressure and constant humidity, speed of sound waves in gas is directly proportional to the square root of its absolute temperature.

$$v = \sqrt{\frac{\gamma RT}{M}} \quad \therefore \; v \propto \sqrt{T}$$

The speed of sound in a gas does not depend on the pressure variation.

10. **Principle of Superposition :** When a particle of medium comes under the influence of two or more waves simultaneously, its net displacement is the vector sum of displacement that could occur under the influence of the individual waves.

11. **Stationary Waves :** When two waves having same amplitude and frequency and travelling in mutually opposite directions are superposed the resultant wave formed loses the property of propagation. Such a wave is called a stationary wave.

Equation of stationary wave : $y = -2A \sin kx \cos \omega t$

Amplitude of stationary wave : $2A \sin kx$

Position of nodes in stationary wave $x_n = \frac{n\lambda}{2}$

where, $n = 1, 2, 3.....$ At all these points the amplitude is zero.

Position of antinodes in stationary waves,

$x_n = (2n - 1)\frac{\lambda}{4}$ where , $n = 1, 2, 3,....$

The amplitude of all these points is 2A.

12. Frequencies corresponding to different normal modes of vibration in a stretched string of length L fixed at both the ends are given by,

$f_n = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$ where $n = 1, 2, 3......$

13. In a closed pipe the values of possible wavelengths required for stationary wave pattern are given by.

$\lambda_n = \frac{4L}{(2n-1)}$ and possible frequencies, $f_n = (2n - 1)\frac{v}{4L} = (2n - 1)f_1$

where, $n = 1, 2, 3,.....$ and L = length of pipe.

In a closed pipe only odd harmonics like f_1, $3f_1$, $5f_1$, ... are possible.

14. In an open pipe the values of possible wavelength required for stationary waves are given by,

$\lambda_n = \frac{2L}{n}$ and possible frequencies, $f_n = \frac{nv}{2L} = nf_1$ where, $n = 1, 2, 3,$ ...... and L = length of pipe.

In open pipe of the harmonics like f_1, $2f_1$, $3f_1$, ... are possible.

15. **Beat :** The phenomenon of the loudness of sound becoming maximum periodically due to superposition of two sound waves of equal amplitude and slightly different frequencies is called the 'beats'.

Number of beats produced in unit time $= f_1 - f_2$

16. **Doppler Effect :** Whenever there is a relative motion between a source of sound and a listener with respect to the medium in which the waves are propagating the frequency of sound experienced by the listener is different from that which is emitted by the source. This phenomenon is called Doppler effect.

Frequency listened by the listener, $f_L = \frac{v \pm v_L}{v \pm v_S} f_S$

Where, v = velocity of sound, v_L = velocity of a listener,

v_S = velocity of a source, f_S = frequency of sound emitted by the source.

EXERCISES

Choose the correct option from the given options :

1. Mechanical waves carry ....

(A) energy (B) matter

(C) both energy and matter (D) neither energy nor matter.

2. A tuning fork makes 256 vibrations per second in air. When the velocity of sound is 330 m/s, then wavelength of the wave emitted is ....

(A) 0.56 cm (B) 0.89 m (C) 1.11 m (D) 1.29 m

3. When a sound wave of frequency 300 Hz passes through a medium, the maximum displacement of a particle of the medium is 0.1 cm. The maximum velocity of the particle is equal to ....

(A) 60π cm/s (B) 30π cm/s (C) 30 cm/s (D) 60 cm/s

4. The speed of wave of frequency 500 Hz is 360 m s^{-1}. The minimum distance between two particles on it, having phase difference of 60° is ....

(A) 0.23 m (B) 0.12 m (C) 8.33 m (D) 60 m

5. If the speed of the wave shown in the Figure is 330 m/s in the given medium, then the equation of the wave propagating in the positive x-direction will be ....

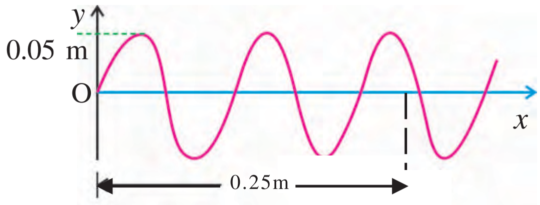


Figure 8.18

(A) $y = 0.05 \sin 2\pi\ (4000\ t - 12.5x)$ m

(B) $y = 0.05 \sin 2\pi\ (4000\ t - 122.5x)$ m

(C) $y = 0.05 \sin 2\pi\ (3300\ t - 10x)$ m

(D) $y = 0.05 \sin 2\pi\ (3300\ t - 10t)$ m

6. The equation $y = A \sin^2 (kx - \omega t)$ represents a wave with amplitude ...... and frequency ....

(A) A, $\omega/2\pi$ (B) $\frac{A}{2}, \frac{\omega}{\pi}$ (C) 2A, $\frac{\omega}{4\pi}$ (D) $\sqrt{A}, \frac{\omega}{2\pi}$

7. Two pulse travels in mutually opposite directions in a string with a speed of 2.5 cm/s as shown in figure. Initially (at $t = 0$) the pulses are 10 cm apart. What will be the state of string after two seconds ?

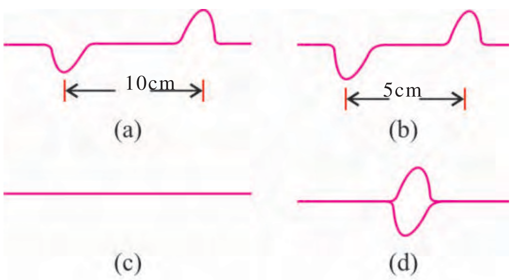


Figure 8.19

8. The speed of the component waves of a stationary wave represented by $y = 10 \sin (100\ t) \cos (0.01x)$ is ....
Where, x and y are in metre and t is in second.

(A) 1 m s^{-1} (B) 10^2 m s^{-1} (C) 10^3 m s^{-1} (D) 10^4 m s^{-1}

9. The mass of 7 m long string is 0.035 kg. If the tension in the string is 60.5 N, then the speed of wave on string will be ....

(A) 77 m s^{-1} (B) 102 m s^{-1} (C) 110 m s^{-1} (D) 165 m s^{-1}

10. If the maximum intensity of the beat produced by the superposition of two waves is x times the intensity of superposing wave then x = ...

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) 4

11. Two waves of wavelengths 2.00 m and 2.02 m superpose with each other to produce beats in 1 s. If the speed of both waves is the same, their same speed is ....

(A) 400 m/s (B) 402 m/s (C) 404 m/s (D) 406 m/s

12. The speed of the component waves of a stationary wave is 1200 m/s. If the distance between consecutive antinode and node is 1 m, then frequency of standing wave will be ....

(A) 300 Hz (B) 400 Hz (C) 600 Hz (D) 1200 Hz

13. Suppose the listener and sound source both are approaching each other with speed of 50 m/s on a straight path. If the v frequency listened by listener is 440 Hz, what is the frequency of the wave produced by the source ? (speed of wave in air is 340 m/s)

(A) 327 s^{-1} (B) 367 s^{-1} (C) 390 s^{-1} (D) 591 s^{-1}

14. The fundamental frequency of the air column in a closed pipe is 512 Hz. If the pipe is open from both the ends, the fundamental frequency will be .... Hz.

(A) 1024 (B) 512 (C) 256 (D) 128

15. The air column in a closed pipe experiences first resonance with a tuning fork of frequency 264 Hz. If the length of the air column in the closed pipe is ..... cm. The speed of the sound in air is 330 m/s.

(A) 31.25 (B) 62.50 (C) 93.75 (D) 125

16. When the temperature of an ideal gas is increased by 600 K, the velocity of sound in the gas becomes $\sqrt{3}$ times the initial velocity in it. The initial temperature of the gas is ....

(A) −73 °C (B) 27 °C (C) 127 °C (D) 327 °C

17. Beats are produced by two waves given by y_1 = A sin (2000π)t (m) and y_2 = A sin (2008π)t (m). The number of beats heard per second is ....

(A) 0 (B) 1 (C) 4 (D) 8

18. A source of sound is moving towards a stationary listener with 1/10 of the speed of sound. The ratio of apparent to real frequency is ....

(A) 10/9 (B) 11/10 (C) $(11/10)^2$ (D) $(9/10)^2$

19. A transverse wave is described by the equation $y = \text{A} \sin 2\pi \left(\frac{t}{\text{T}} - \frac{x}{\lambda}\right)$. For which wavelength of a wave, maximum particle velocity is two times the wave velocity ?

(A) $\lambda = \frac{\pi \text{A}}{4}$ (B) $\lambda = \frac{\pi \text{A}}{2}$ (C) $\lambda = \pi \text{A}$ (D) $\lambda = 2\pi \text{A}$

20. The temperature at which speed of sound in air becomes double of its value at 0°C is ....

(A) 273 K (B) 546 K (C) 1092 K (D) 0 K

ANSWERS

1. (A)	**2.** (D)	**3.** (A)	**4.** (B)	**5.** (C)	**6.** (B)
7. (C)	**8.** (D)	**9.** (C)	**10.** (D)	**11.** (C)	**12.** (A)
13. (A)	**14.** (A)	**15.** (A)	**16.** (B)	**17.** (C)	**18.** (A)
19. (C)	**20.** (C)				

Answer the following questions in short :

1. Give the definition of wave intensity and give its SI unit.

2. What is angular wave number of a wave ?

3. What is the distance travelled by the progressive wave if its wavelength is λ and frequency is f ?

4. Which characteristics of a medium are required for the propagation of mechanical wave ?

5. What is pressure wave ?

6. How the wave speed is changing with change in the temperature of a medium ?

7. What will be change in the speed of a wave in wire if the tension in wire increased four times ?

8. What will be the effect on the speed of a wave if the pressure of the medium will change ?

9. The wave equation of a wave is $y = 5 \sin (0.01x - 2t)$. Where x and y are in cm. What is the speed of a wave ?

10. What will be the change in the phase of a wave, if the wave on the string is reflected from the rigid support ?

11. What is the amplitude of node and antinode in a stationary wave ?

12. What is the distance between consecutive antinode in a stationary wave if the distance between consecutive node and antinode is 5 cm ?

13. In a closed pipe fundamental frequency is 300 Hz. What will be the frequency of second overtone ?

14. Frequency of the source of sound is 440 Hz. If the relative velocity of source and listener is zero then which frequency will be listened by a listener ?

15. What is a beat ?

Answer the following questions :

1. Explain the classification of the waves. Give the example of each wave.

2. Explain wavelength, wave number and frequency of a wave.

3. With the help of dimensional analysis, obtain the expression for the wave speed propagating in the string kept under tension.

4. Explain the propagation of sound waves in the air.

5. Write the Newton's formula for speed of a wave in air. Expain Laplace correction in Newton's formula.

6. Obtain the one dimension wave equation $y = A \sin(\omega t - kx)$ for the wave propagating in the direction of increasing value of x.
7. Write the superposition principle for the waves and explain it.
8. What are stationary waves ? Obtain the expression for the stationary wave in case of string fixed at its two ends.
9. Show that in a closed pipe the harmonics are possible only for odd multiples of fundamental frequency.
10. What is Doppler effect ? When the source of sound is stationary and listener is moving towards the source, obtain the expression for the wavelength of wave travelling towards the listner.

Solve the following problems :

1. In case of the progressive harmonic waves, prove that the ratio of the instantaneous velocity of any particle of the medium to the wave speed is equal to the negative of the slope of the waveform at that point at that instant.
2. Two types of waves, transverse (S) and longitudinal (P) are produced in the earth during an earthquake. The speed of the S wave is approximately 4.0 km/s and that of the P wave is 8.0 km/s. In a seismograph, recording the earthquake the P wave is recorded 4 min earlier than the S wave. Assuming that both types of waves travel on straight line, find the distance of the origin of the quake from the seismograph. [**Ans. :** Approximately 1920 km]
3. The amplitude of the progressive harmonic wave is 10 m. During the wave propagation, the displacement of a particle which is at a distance of 2 m from the origin is 5 m after 2 s. Another particle which is at 16 m from origin has displacement of $5\sqrt{3}$ m in 8 s. Find the angular frequency and wave vector of a wave. [**Ans. :** $\omega = \pi/8$ rad/s, $k = \pi/24$ rad/m]
4. The equation for a wave travelling in x–direction on a string is, $y = 3 \sin[(3.14x - (314)t]$. Where x is in cm and t is in second.
   (i) Find the maximum velocity of a particle of the string.
   (ii) Find the acceleration of a particle at $x = 6.0$ cm at time $= t = 0.11$ s.
   [**Ans. :** Maximum velocity = 9.4 m/s, $a = 0$]
5. At 0 °C temperature, a source of sound of frequency 250 Hz emits sound waves of wavelength 1.32 m. What will be the increase in wavelength at 27 °C ? [**Ans. :** 0.06 m]
6. At what temperature the hydrogen gas will have the speed of sound waves in it will be equal to the speed of sound in oxygen at 1200 °C ? The density of oxygen is 16 times that of hydrogen. [**Ans. :**–180.9 °C]
7. The length of a sonometer wire between its fixed ends is 110 cm. Where should the two bridges S_1 and S_2 be placed in between the ends so as to divide the wire into 3 segments whose fundamental frequencies are $f_1 : f_2 : f_3 = 1 : 2 : 3$? [**Ans. :** $L_1 = 60$ cm, $L_2 = 30$ cm, $L_3 = 20$ cm]

8. A wire having a linear mass density 0.05 g/cm is stretched betwen two rigid supports with a tension of 450 N. The wire resonates at a frequency of 420 Hz. The next higher frequency at which the same wire resonates is 490 Hz. Find the length of the wire. [**Ans. :** 2.1 m]

9. The length of a string is 100 cm. The frequencies of two consecutive harmonics formed on the string are 300 Hz and 400 Hz respectively. The maximum amplitude is 10 cm when the string oscillates with its fundamental frequency. Write the equation of the stationary wave in this case.

[**Ans. :** $y = -10\sin\left(\frac{\pi x}{100}\right)\cos(200\pi)t$ (cm)]

10. Find the difference of apparent frequencies of the sound of a car horn heard by a stationary listener when the car is moving towards and away from the listener with a speed 54 km/4. The frequency of sound emitted by the horn is 500 Hz and speed of sound in air is 340 m/s. [**Ans. :** 44.2 Hz]

11. The whistle of an engine, approaching a hill with a speed of 10 m/s produces sound of frequency 660 Hz. Find the frequency experienced by the driver of the sound reflected from the hill. The speed of sound in air is 340 m/s.

[**Ans. :** 700 Hz]

• • •

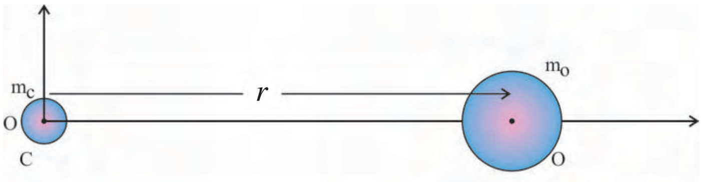
Meghnad Saha (1893-1956)

Meghnad Saha was born on October 6, 1893 in Sheoratali, a village in the District of Dacca (now in Bangladesh). In 1911, he came to Calcutta to study in Presidency College. He came to be recognised as a scientist of substance. In 1920, he went to England to prove his theory–equation of reaction–before the global scientific community. This later became Saha's Thermo Ionization Equation. In 1927, Meghnad was elected as a fellow of London's Royal Society. He invented an instrument to measure the weight and pressure of solar rays. The lasting memorial to him is the 'Saha Institute of Nuclear Physics' founded in 1943 in Calcutta. Saha passed away on February 16, 1956.

SOLUTION

CHAPTER 1

1.



Figure

Here the origin is taken on the centre of carbon (C).

r = distance of oxygen from carbon = 1.130×10^{-10} m,

m_O = mass of oxygen = 16 g mol^{-1}, m_C = mass of carbon = 12 g mol^{-1}

r_C = distance of carbon from origin = 0, r_O = distance of oxygen from origin = r = 1.130×10^{-1}m

$$\therefore\ r_{cm} = \frac{m_C r_C + m_O r_O}{m_C + m_O}$$

2. Velocity of centre of mass $\vec{v}_{cm} = \dfrac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3}{m_1 + m_2 + m_3}$

3. Here for car m_1 = 1000 kg, a_1 = 4.0 m s^{-2}, initial speed v_{0_1} = 0 m s^{-1}

For truck m_2 = 2000 kg, a_2 = 0 m s^{-2}, $v_{0_2} = v_2$ = 8.0 m s^{-1}

After 3 sec, the speed of car $v_1 = v_{0_1} + a_1 t$

After 3 secs the distance travelled by car $d_1 = v_{0_1} t + \frac{1}{2} a_1 t^2$

The distance travelled by truck in 3 sec. $d_2 = v_2 t$ ($\because\ a_2 = 0$)

(a) The distance of centre of mass of the system of car-truck is

$$d_{cm} = \frac{m_1 d_1 + m_2 d_2}{m_1 + m_2}$$

(b) In one dimension $Mv_{cm} = m_1 v_1 + m_2 v_2$

$$\therefore\ v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \quad (\because M = m_1 + m_2)$$

4. At t = 0 sec. $x_1 = -15\ m$, $x_2 = 15\ m$

m_1 = 40 kg, m_2 = 20 kg

$$\therefore\ x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

As the centre of mass is remaining stationary, x_{cm} = const. Hence find x_2 from the values of x_1 and x_{cm} for t = 2, 4 ,6 sec. At t = 0, the cat and dog are at rest.

$\therefore\ v_1 = v_2 = 0 \Rightarrow p_1 = p_2 = 0$

$\Rightarrow p = p_1 + p_2 = 0$

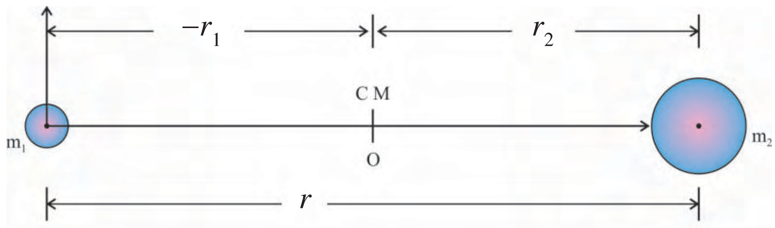
At $t = 2$ s

$$v_1 = \frac{x_1(2\text{ s}) - x_1(0\text{ s})}{2\text{ s}} = \frac{\Delta x}{\Delta t},\ v_2 = \frac{x_2(2\text{ s}) - x_2(0\text{ s})}{2\text{ s}}$$

Hence, $p_1 = m_1 v_1$, $p_2 = m_2 v_2$ and $p = p_1 + p_2$

Similarly, repeat calculations for $t = 4$ s and $t = 6$ s.

5.



Figure

In figure, the origin is taken at centre of mass.

$\therefore$ Position of m_1 from origin $= -r_1$, Position of m_2 from origin $= r_2$

$$\therefore\ r_{cm} = 0 = \frac{m_1(-r_1) + m_2 r_2}{m_1 + m_2} \qquad \therefore\ m_1 r_1 = m_2 r_2$$

$$\therefore\ \frac{m_1}{m_2} = \frac{r_2}{r_1} \tag{1}$$

Performing union in the denominator $\dfrac{m_1}{m_1 + m_2} = \dfrac{r_1}{r_1 + r_2} = \dfrac{r_2}{r}$

$$(\because\ r = r_1 + r_2)\ \therefore\ r_2 = r\left[\frac{m_1}{m_1 + m_2}\right]$$

Performing union in numerator in equation (1) $\dfrac{m_1 + m_2}{m_2} = \dfrac{r_1 + r_2}{r_1} = \dfrac{r}{r_1}$

$$\therefore\ r_1 = r\left[\frac{m_2}{m_1 + m_2}\right]$$

6. Here, the centre of mass of the system made up of three spheres is

$$\vec{r}_{cm} = \frac{m\vec{r}_{cm1} + m\vec{r}_{cm2} + m\vec{r}_{cm3}}{m + m + m},$$ Where $\vec{r}_{cm1}$ = centre of mass of sphere 1, etc.

7. Here, the density of sphere of radius R is ρ. Hence, the mass of original sphere

$$M = \rho V = \rho \times \frac{4}{3}\pi R^3 \tag{i}$$

The mass of small sphere of radius 'a' is $m_1 = \rho \times \dfrac{4}{3}\pi a^3$ (ii)

Hence, the mass of the remaining sphere after removing small sphere of radius 'a' from the original sphere of radius 'R' is $m_2 = M - m_1$.

$$\therefore\ m_2 = \frac{4}{3}\pi\rho\ (R^3 - a^3) \tag{iii}$$

The centre of mass of original sphere $\vec{r}_{cm} = (0, 0, 0)$

The centre of mass of small sphere of radius 'a', $\vec{r}_1 = (b, 0, 0)$

The remaining sphere has symmetry about X-axis, but no symmetry about Y and Z axes. Hence, the centre of mass of remaining sphere is, say $\vec{r}_2 = (-x, 0, 0)$

The sphere of radius R is made up of small sphere of radius 'a' and the remaining sphere (without small sphere). Hence, $M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2$.

$\therefore$ M(0, 0, 0) = m_1(b, 0, 0) + m_2($-x$, 0, 0). Comparing x co-ordinates

$$M(0) = m_1 b - m_2 x \;\therefore\; x = \frac{m_1}{m_2} b \qquad (iv)$$

Using results (ii) and (iii), we get x.

8. From the figure, the masses of the three particles, the positions and forces acting on them during steady positions are

$m_1 = 4.0$ kg, $\quad \vec{r}_1 = (-2, 3)\ m$, $\quad \vec{F}_1 = (-6, 0)$ N

$m_2 = 8.0$ kg, $\quad \vec{r}_2 = (4, 2)\ m$, $\quad \vec{F}_2 = (12\cos 45^o, 12\sin 45^o)$ N

$m_3 = 4.0$ kg, $\quad \vec{r}_3 = (1, -2)\ m$, $\quad \vec{F}_3 = (14, 0)$ N

$$\therefore \vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

According to the Newton's second law $\vec{F} = M\vec{a}_{cm}$, $M = m_1 + m_2 + m_3$

$$\therefore \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = M\vec{a}_{cm} \qquad \therefore \vec{a}_{cm} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M}$$

$$\therefore \vec{a}_{cm} = (a_{x_{cm}}, a_{y_{cm}})$$

Hence the magnitude of acceleration $|\vec{a}_{cm}| = \sqrt{(a_{xcm})^2 + (a_{ycm})^2}$

and the direction of acceleration with X-axis is $\theta = tan^{-1}\left(\frac{a_{ycm}}{a_{xcm}}\right)$ = ...

9. From the figure, the centre of mass of the original plate of uniform density 'ρ' and radius 'R' is $\vec{r}_{cm} = (0, 0)$ (1)

The centre of mass of plate of radius $\frac{R}{2}$ is $\vec{r}_{cm_1} = \left(\frac{R}{2}, 0\right)$ (2)

When the plate of radius $\frac{R}{2}$ is cut from the plate of radius R, the remaining plate has symmetry about X-axis, but no symmetry about Y-axis. Hence the centre of mass of remaining plate must be away from the origin along X-axis as say $(-x)$. $\therefore\ \vec{r}_{cm_2} = (-x, 0)$ (3)

The original plate is made up plate of radius $\frac{R}{2}$ and remaining plate

$$\therefore\ \vec{r}_{cm} = \frac{M_1 \vec{r}_{cm1} + M_2 \vec{r}_{cm2}}{M_1 + M_2} \quad (4)$$

Where M_1 = Mass of plate of radius $\frac{R}{2}$ $\therefore\ M_1 = \pi\left(\frac{R}{2}\right)^2 t\rho$

M_2 = Mass of remaining plate $= \pi R^2 t\rho - M_1 = \pi R^2 t\rho - \pi\left(\frac{R}{2}\right)^2 t\rho$

$$M_2 = \pi t\rho\left[R^2 - \left(\frac{R}{2}\right)^2\right]$$

Where ρ = Density of plate, t = thickness of plate

Hence, from equation (4) calculate $\vec{r}_{cm2}$.

CHAPTER 2

1. Using equation $\theta = \left(\frac{\omega + \omega_0}{2}\right)t$ find ω_0. Substituting ω_0 in the equation $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ find α.

2. Substituting values $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ find α. Now from $\theta = \frac{\omega^2 + \omega_0^2}{2\alpha}$ find θ and represent θ in rotations. (2π rad = 1 rotation)

3. Using $\alpha = \frac{\omega - \omega_0}{t}$, find α. Now $I = m\,r^2$, using $\tau = I\alpha$, find τ from $\theta = \frac{\omega^2 + \omega_0^2}{2\alpha}$ find θ. Now work $= \tau \cdot \theta$

4. Use $\vec{l} = \vec{r} \times \vec{p}$, $\vec{r} = 4\hat{i} + 6\hat{j} + 12\hat{k}$ and $\vec{p} = m\vec{v} = 50\,(2\hat{i} + 3\hat{j} + 6\hat{k})$

5. Linear acceleration for a body rolling down the slope is $a = \frac{g\sin\theta}{\left[1 + \frac{K^2}{R^2}\right]}$

Substituting K = R the radius of gyration for hollow cylinder obtain a.

6. Moment of inertia of the system $I_z = I_{1z} + I_{2z}$. I_{1z} = Moment of inertia of the object of 100 kg relative to z axis I_{2z} = moment of inertia of the object of 200 kg relative to z axis. As the distances are relative to z axis, z coordinate is not taken in to calculation.

$I_z = I_x + I_y = m(x^2 + y^2)$ (1)

position vectors for the objects of 100 kg and 200 kg are (2, 4, 6) and (3, 5, 7) respectively.

$\therefore\ I_{1z} = I_{1x_1} + I_{2y_1} = 100\ (x^2_{\ 1} + y^2_{\ 1}),\ I_{2z} = I_{1x_2} + I_{2y_2} = 200\ (x^2_{\ 2} + y^2_{\ 2})$

Substitute in (1)

7. Using $K = \sqrt{\frac{2}{5}}R$ for solid sphere in $v^2 = \dfrac{g\sin\theta}{\left[1 + \dfrac{K^2}{R^2}\right]}$ find v.

Now using $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. Calculate rotational kinetic energy. $(\frac{1}{2}I\omega^2)$

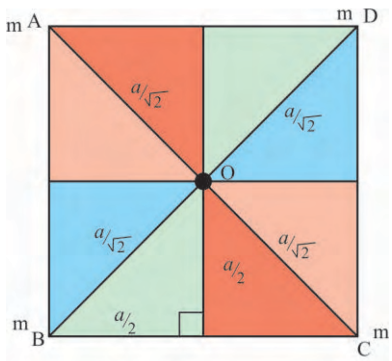
8. Consider Earth as solid sphere and taking its moment of inertia $I = \frac{2}{5}MR^2$ and $\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 3600}$ substitute in $L = I\omega$ and obtain L.

9. $I_1 = I_C + Md_1^{\ 2} \qquad \therefore\ I_C = I_1 - Md_1^{\ 2}$

Now $I_2 = I_C + Md_2^{\ 2} = I_1 - Md_1^{\ 2} + Md_2^{\ 2}$

$= I_1 + M(d_2^{\ 2} - d_1^{\ 2})$

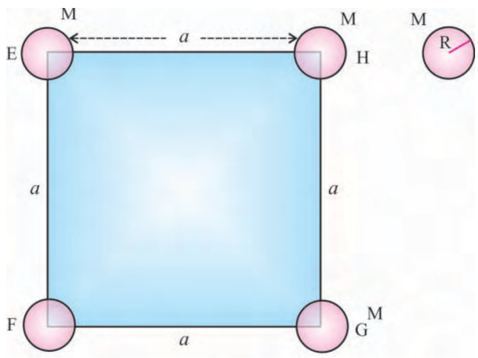
10. From the figure the moment of inertia I of the system about the axis passing through O.



$$I = \frac{ma^2}{2} + \frac{ma^2}{2} + \frac{ma^2}{2} + \frac{ma^2}{2} + = 2ma^2$$

11. Moment of inertia of the sphere about the axis passing through its centre is $I_C = \frac{2}{5}MR^2$

From the figure moment of inertia of the system about the axis EF



$I = I_E + I_F + I_G + I_H$

Using $I = I_C + Md^2$

$I_E = \frac{2}{5}MR^2;\ I_F = \frac{2}{5}MR^2;$

$$I_G = \frac{2}{5}MR^2 + Ma^2; \; I_H = \frac{2}{5}MR^2 + Ma^2$$

$$\therefore \; I = \frac{2}{5}MR^2 + \frac{2}{5}MR^2 + \frac{2}{5}MR^2 + ma^2 + .\frac{2}{5}MR^2 + ma^2$$

$$= 2\left(\frac{4}{5}MR^2 + Ma^2\right)$$

12. $r_1 = 0$, $r_2 = 2m$, $r_3 = 4m$, $r_4 = 6m$, $m_1 = 1$ kg, $m_2 = 2$ kg, $m_3 = 3$ kg, $m_4 = 4$ kg

Now, $I_{AB} = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2$

13. Total kinetic energy = Linear kinetic energy + Rotational kinetic energy

$$= \frac{1}{2}mv^2 + \frac{1}{2} I\omega^2$$

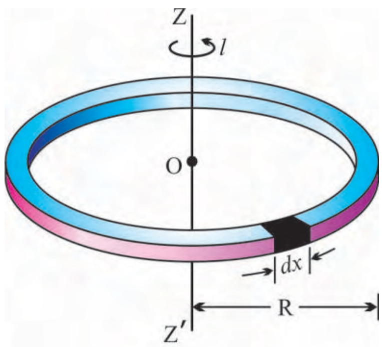
for disc $I = \frac{mr^2}{2}$ substituting $\omega = \frac{v}{r}$ $(\because v = r\omega)$

Total kinetic energy $= \frac{1}{2}mv^2 + \frac{1}{2}\frac{mr^2}{2}\frac{v^2}{r^2} = \frac{3}{4}mv^2$

Rotational kinetic energy $= \frac{1}{4}mv^2$

The fraction of total kinetic energy in the form of rotational kinetic energy $= \dfrac{\frac{1}{4}mv^2}{\frac{3}{4}mv^2} = \frac{1}{3}$

14. To find moment of inertia of thin circular ring or circular wire about an axis passing through its centre and perpendicular to its plane and radius of gyration, consider a thin ring with mass M and radius R as shown in the figure. Length of the ring l that is the circumference of the ring is $2\pi R$.



Mass per unit length of this ring

$$\lambda = \frac{\text{Mass of the ring}}{\text{Length of the ring}} = \frac{M}{2\pi R}$$

Mass of the element of the length dx as shown in the figure $= \lambda \cdot dx = \frac{M}{2\pi R}dx$

If dI is the moment of inertia about the axis ZZ'.

dI = (mass of the element) (perpendicular distance from ZZ' axis)

$$= \left(\frac{M}{2\pi R} \cdot dx\right)(R^2)$$

$$dI = \frac{M}{2\pi} R \cdot dx \qquad (1)$$

For the moment of inertia I of the ring as a whole about axis ZZ' integrate equation (1) in the interval from $x = 0$ to $x = 2\pi R$.

$$\therefore I = \int dI = \int_0^{2\pi R} \frac{M}{2\pi} R \cdot dx$$

$$\therefore I = \frac{M}{2\pi} R \int_0^{2\pi R} dx = \frac{M}{2\pi} R [x]_0^{2\pi R} = \frac{M}{2\pi} R[2\pi R - 0] \quad I = MR^2 \qquad (2)$$

Comparing equation (2) with $I = MK^2$, $K^2 = R^2$, Radius of gyration $K = R$

15. The vector sum of the forces acting on the light rod,

$$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4} + \vec{F_5} \quad (\vec{F} \text{ is the resultant force})$$

$$\vec{F} = \vec{F_1}\hat{j} + \vec{F_2}\hat{j} + \vec{F_3}(-\hat{j}) + \vec{F_4}\hat{j} + \vec{F_5}(-\hat{j})$$

Now, moment of force $\vec{F}$ relative to point A = vector sum of the moments of component forces.

$\therefore F \cdot x = [F_1 \times 0] + [F_2 \times x_1] - [F_3 \times (x_1 + x_2)] + [F_4 \times (x_1 + x_2 + x_3) - [F_5 \times (x_1 + x_2 + x_3 + x_4)]$

$$\therefore x = \frac{x_1F_2 - (x_1 + x_2)F_3 + (x_1 + x_2 + x_3)F_4 - (x_1 + x_2 + x_3 + x_4)F_5}{F_1 + F_2 + F_4 - F_3 - F_5}$$

CHAPTER 3

1. If two forces become equal at distance x from the centre of the Earth,

$$\frac{GM_e m}{x^2} = \frac{GM_s m}{(r - x)^2}, \qquad M_e = \text{Mass of the Earth,}$$

M_s = Mass of the sun. r = Distance between the sun and the Earth. From this find x.

2. M_e = Volume × Density = $\left(\frac{4}{3}\pi R_e^{\,3}\right)(\rho)$

$$\therefore g = \frac{GM_e}{R_e^{\,2}} = \frac{4}{3}\pi G\rho R_e. \text{ Hence find } g.$$

3. $$\left\{\begin{matrix}\text{The required centripetal} \\ \text{force for the Earth's} \\ \text{circular motion} \\ \dfrac{M_e v_0^2}{r}\end{matrix}\right\} = \left\{\begin{matrix}\text{The gravitational force} \\ \text{on the Earth by} \\ \text{the Sun} \\ \dfrac{GM_s M_e}{r^2}\end{matrix}\right\}$$

$$\therefore\ M_s = \frac{r v_0^2}{G}$$

4. For the circular motion of the satellite,

$$v_0 = \sqrt{\frac{GM_e}{r}} = \sqrt{\frac{GM_e}{2R_e}} \quad (\because r = R_e + R_e = 2R_e)$$

Find v_0 from this. Now $T^2 = \left(\dfrac{4\pi^2}{GM_e}\right) r^3$. From this find T

5. For circular motion of satellite $mv^2/r = GM_e\, m/r^2$

$\therefore$ Kinetic energy of satellite $\dfrac{1}{2}mv^2 = \dfrac{GM_e m}{2r}$

But potential energy $= \dfrac{-GM_e m}{r}$

$\therefore$ Total energy = kinetic energy + potential energy $= \dfrac{-GM_e m}{2r}$

$\therefore$ Escape energy $= \dfrac{GM_e m}{2r}$

$\therefore\ \dfrac{1}{2}mv_e^2 = \dfrac{GM_e m}{2r}$. From this find v_e.

6. For the circular motion of the satellite, $\dfrac{mv^2}{R_e} = \dfrac{GM_e m}{R_e^2} = (g)m$ (1)

$(\because\ g = \dfrac{GM_e}{R_e^2})$ $\quad\therefore\ v^2 = gR_e$. But $v = \dfrac{2\pi R_e}{T}$

Put this value in equation (1) and find T.

7. For circular motion of the satellite, $\dfrac{mv_0^2}{R_e} = \dfrac{GM_e m}{R_e^2}$ $\quad\therefore\ v_0 = \sqrt{\dfrac{GM_e}{R_e}}$

For the object lying on the surface of the Earth, $v_e = \sqrt{\dfrac{2GM_e}{R_e}}$. Find $\dfrac{v_0}{v_e}$

8. At the given point the total energy $= \left[-\dfrac{GM_1 m}{d/2}\right] + \left[\dfrac{-GM_2 m}{d/2}\right]$

$= \dfrac{-2G(M_1+M_2)m}{d}$ $\quad\therefore$ Escape energy $= \dfrac{2G(M_1+M_2)m}{d}$

If the escape velocity is v_e, then $\dfrac{1}{2}mv_e^2 = \dfrac{2G(M_1+M_2)m}{d}$. Hence, find v_e

9. In this special case, the circular motion is governed by,

$$\left(\text{Centripetal force } \frac{mv^2}{r}\right) = \left(\text{Gravitational force } \frac{GMm}{r^{5/2}}\right)$$

Also put $v = \frac{2\pi r}{T}$. Hence, find T^2.

CHAPTER 4

1. Here, weight of wire = tensile force = ldg and breacking stress =

$$\frac{\text{Tensile force}}{\text{Area}} = ldg \quad \therefore\ L = \frac{\text{Breaking stress}}{dg}$$

2. If increase in lengths of AB, BC and CD wires are Δl_{AB}, Δl_{BC} and Δl_{CD}, find these increments using

$\Delta l = \frac{Fl}{AY}$, Displacement of B = Δl_{AB}, Displacement of C = $\Delta l_{AB} + \Delta l_{BC}$

and Displacement of D = $\Delta l_{AB} + \Delta l_{BC} + \Delta l_{CD}$.

3. Centripetal force necessary for circular motion is supplied by restoring force.

$Y = \frac{FL}{A\Delta l}$ $\therefore$ $F = \frac{YA\Delta L}{L}$ and $F = \frac{mv^2}{L} = \frac{m\omega^2 L^2}{L}$ compose these two values of F.

4. Draw F.B.D. for both the masses and calculate tension T.

Here, Stress = $\frac{T}{A}$ and $\frac{\Delta l}{l} = \frac{\text{Stress}}{Y}$

5. First ditermine Δl using $Y = \frac{Fl}{A\Delta L}$; now use illustration 3.

Use $U = \frac{1}{2} Y \times \text{stress} \times \text{strain} \times \text{volume}$.

6. $\Delta l = l \propto \Delta t \ \therefore\ \frac{\Delta l}{l} = \propto \Delta t$

Now use $Y = \frac{F}{A}\frac{l}{\Delta t}$; here F is tension. Now find F.

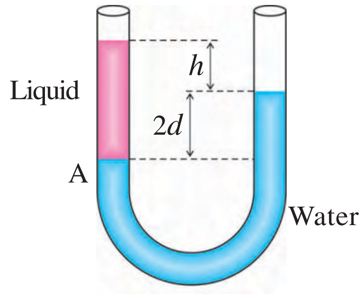
CHAPTER 5

1. Find velocity of water coming out of nozzle using $A_1 v_1 = A_2 v_2$

Now for vertical motion $y = \frac{1}{2} g t^2$, $y = 1$ m, for horizontal motion $x = v_2 t$

$$\therefore\ y = \frac{1}{2} g \left(\frac{x}{v_2}\right)^2 \quad \therefore\ x = \sqrt{\frac{2 y v_2^2}{g}}$$

2. Pressure at A = Pressure at B



$\therefore$ $(h + 2d)\rho_l g + P_a = P_a + 1(2d)g$

Now, find ρ_l.

3. For horizontal flow

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\therefore P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_2^2)$$

$\therefore$ $\rho_{Hg}\, g(h_1 - h_2) = \frac{1}{2}\rho_{water}\,(r_2^2 - r_1^2)$ insert other values to get h_2.

4. Work $= T\Delta A = T2\pi\ (r_2^2 - r_1^2)$

5. $T = \dfrac{rh\rho g}{2cos\theta}$

$\therefore$ $h = \dfrac{2Tcos\theta}{r\rho g}$ Use this formula to calculate heights of water in both the arms. Then find the differences.

6. $\eta = \dfrac{2}{9}\dfrac{v^2}{vt}(\rho - \rho_0)g$

Here, constant velocity of bubble is the terminal velocity.

7. and **8.** Use equation given in hint.

9. Find P_i usng $P_i - P_o = \dfrac{4T}{R}$ $P_o = 10^5$ Pa

Now for isothermal change $P_iV = P_i'\dfrac{V}{8}$

Find P_i', No $P_i' - P_o' = \dfrac{4T}{R'}$ take $R' = \dfrac{R}{2}$ and calculate P_o'.

CHAPTER 6

1. $m = 200$ g, $\Delta T = T_f - T_i$, $C = 0.215$ cal g^{-1} C^{o-1}, $Q = mC\Delta T$

and $H_C = \dfrac{Q}{\Delta T}$

2. (*a*) 32 g O_2 = 1 mole

$\therefore$ 10 g $O_2 = \dfrac{10}{32} = \dfrac{5}{6}$ mole

$\therefore$ $\mu = \dfrac{5}{6}$ mole

$P = 3 \times 10^5$ N m^{-2}, $T = 273 + 10 = 283$ K

From the ideal gas, state equation,

$PV_1 = \mu RT_1 \Rightarrow V_1 = \dfrac{\mu RT_1}{P}$

and $V_2 = 10$ L $= 10^{-2}$ m^3

Hence, the work done by the gas

$W = P(V_2 - V_1)$

(*b*) As O_2 is diatomic rigid rotator,

$$C_V = \frac{5}{2}R$$

and $PV_2 = \mu RT_2 \Rightarrow T_2 = \dfrac{PV_2}{\mu R}$

$\therefore\ \Delta E_{int} = C_V\,(T_2 - T_1)$

(*c*) $\Delta E_{int} = Q - W$

$\therefore\ Q = \Delta E_{int} + W$

3. Here $T_2 = 300$ K, $\eta = 40\ \% = 0.4$, $\eta = 1 - \dfrac{T_2}{T_1}$, Hence calculate T_1.

Keeping T_1 = constant, $\eta' = 50\% = 0.5$, then $T_2' = ?$

From, $\eta' = 1 - \dfrac{T_2'}{T_1}$, find T_2'.

4. $T_1 = 500$ K, $T_2 = 375$ K, $Q_1 = 600$ k cal

(i) Efficiency $\eta = 1 - \dfrac{T_2}{T_1}$ (ii) $\dfrac{Q_2}{Q_1} = \dfrac{T_2}{T_1} \Rightarrow Q_2 = \dfrac{T_2}{T_1} \times Q_1$

Hence network done $W = (Q_1 - Q_2) \times 4.2\dfrac{J}{cal}$

(iii) Heat gained back in heat sink is $= Q_2$

5. $T_i = 27\ ^{o}C = 27 + 273 = 300$ K

$P_i = 2$ atm, $\mu = 1$ mol, $\gamma = 1.5$, $V_f = \dfrac{1}{8}V_i$

(*a*) For adiabatic process PV^{γ} = constant

$$\therefore\ P_iV_i^{\gamma} = P_f\,V_f^{\gamma},\ \Rightarrow P_f = P_i\left(\frac{V_i}{V_f}\right)^{\gamma}$$

(*b*) According to the ideal gas state equation $P_iV_i = \mu RT_i$

$$P_fV_f = \mu RT_f\ \therefore\ \frac{P_iV_i}{P_fV_f} = \frac{T_i}{T_f} \Rightarrow T_f = T_i\,\frac{P_fV_f}{P_iV_i}$$

6. For adiabatic process $W = \dfrac{\mu R(T_i - T_f)}{\gamma - 1}$. Here the volume decreases, hence, the work done is negative.

$$\therefore\ W = \frac{-\mu R(T_i - T_f)}{\gamma - 1} = \frac{\mu R(T_f - T_i)}{\gamma - 1}$$

7. According to first law of thermodynamics, $\therefore\ \Delta E_{int} = Q - W$

For closed gas container, the volume is constant $\Rightarrow \Delta V = 0\ \therefore\ W = 0$

$$\Delta E_{int} = Q = \mu C_V\Delta T = \frac{PV}{RT}C_V\Delta T\ \ (\because\ PV = \mu RT,\ \therefore\ \mu = \frac{PV}{RT})$$

$$\therefore\ \Delta T = \frac{QRT}{PVC_V}\ \ (\text{For monoatomic gas } C_V = \frac{3}{2}R)$$

$\therefore$ Final temperature $T_f = T_i + \Delta T$. For ideal gas $P_iV_i = \mu RT_i$ $P_fV_f = \mu RT_f$

$(\because\ V_f = V_i)$

$$\therefore\ \frac{P_f}{P_i} = \frac{T_f}{T_i} \Rightarrow P_f = P_i\frac{T_f}{T_i}$$

8. Here, $\mu = 1$ mol, $\Delta T = 30\ C^o = 30$ K, $V \propto T^{\frac{2}{3}}$

$$\therefore\ V = AT^{\frac{2}{3}},\ A = \text{constant},\ \therefore\ dV = A\frac{2}{3}T^{-\frac{1}{3}}dT$$

$$\text{Hence, } W = \int_T^{T+\Delta T} P\,dV = \int_T^{T+\Delta T}\frac{RT}{V}dV\ (\because PV = \mu RT,\ \therefore\ PV = RT,\ \mu = 1)$$

$$= \int_T^{T+\Delta T}\frac{RT}{AT^{\frac{2}{3}}}A\frac{2}{3}T^{-\frac{1}{3}}dT = \frac{2R}{3}\int_T^{T+\Delta T}dT = \frac{2}{3}R\,[T]_T^{T+\Delta T}$$

$$= \frac{2}{3}R[T + \Delta T - T] \qquad \therefore\ W = \frac{2}{3}R\Delta T$$

9. Here, $P = 1.0$ atm $= 1.01 \times 10^5$ N m^{-2}, $T = 300$ K, $\mu = 2$ mol, $R = 8.31$ J mol^{-1} K^{-1}

For diatomic (rigid rotator) gas $\gamma = \frac{7}{5}$. According to ideal gas state equation

$PV = \mu RT$, $\therefore\ V = \frac{\mu RT}{P}$. For adiabatic process, PV^{γ} = Constant

$$\therefore\ \text{Constant} = P\left(\frac{\mu RT}{P}\right)^{\gamma}$$

10. Here $T_1 = 300$ K, $T_2 = 600$ K, $T_3 = 455$ K for monoatomic gas $f = 3$

For 1 mole gas

$E_{int,\ 1} = \frac{fRT_1}{2}$, $E_{int,\ 2} = \frac{fRT_2}{2}$ and $E_{int,\ 3}$ = Internal energy at point 3 $= \frac{fRT_3}{2}$

Process 1 → 2 : Process is isobaric $\Rightarrow W_1 = 0$

$\therefore\ Q_1 = \Delta E_{int,\ 12} = E_{int,\ 2} - E_{int,\ 1}$

Process 3 → 1 : Process is isobaric

$\therefore\ \Delta E_{int,\ 31} = Q_3 - W_3,\ W_3 = PdV$

But as the volume of the gas is decreasing, W is negative

$\therefore$ $W_3 = -PdV = -\mu R(T_3 - T_1)$ and $\Delta E_{int',\ 31} = E_{int',\ 1} - E_{int',\ 3}$

Hence $Q_3 = \Delta E_{int',\ 31} + W_3$

11. $\eta = 22\% = 0.22$, $Q_1 - Q_2 = 75$ J, $\eta = \dfrac{Q_1 - Q_2}{Q_1} \Rightarrow Q_1 = \dfrac{Q_1 - Q_2}{\eta}$

and $Q_2 = Q_1 - 75$ J

12. Here $Q_1 = 10{,}000$ J, W = 2000 J, $L_C = 5.0 \times 10^4$ J/g

(*a*) Efficiency of engine $\eta = \dfrac{W}{Q_1}$,

(*b*) During each cycle, the heat given into heat sink is $Q_2 = Q_1 - W$,

(*c*) Let '*m*' gram gasoline is used during each cycle.

$$\therefore\ Q_1 = mL_C \ \therefore\ m = \frac{Q_1}{L_C},$$

(*d*) Gasoline used in each cycle = *m* gram

$\therefore$ Gasoline used in 25 cycles per second is $M = 25 \times m$ gram

$\therefore$ Gasoline used in 1 hour = $60 \times 60 \times M$ g/h = ... kg/h

(*e*) Power generated by engine in 1 second = Number of cycles per second × (work done during each cycle)

CHAPTER 7

1. (*a*) T = 3 s, A = 2 cm, $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{3}$, $\phi = 60° = \dfrac{\pi}{3}$

$$\therefore\ y = 2\sin\left(\frac{2\pi}{3}t + \frac{\pi}{3}\right)$$

(*b*) T = 1 min = 60 s, A = 3 cm, $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{60}$, $\phi = -90° = -\dfrac{\pi}{2}$

$$\therefore\ y = 3\cos\left(\frac{\pi}{30}t\right)$$

2. $K = k + 2k + k = 8\ \text{N m}^{-1}$, $T = 2\pi\sqrt{\dfrac{m}{K}} = 0.628$ s

3. Here $F = -kl = -k(l_1 + l_2)$, Also $F_1 = -k_1 l_1 = -k(l_1 + \dfrac{l_1}{n}) \therefore k_1 = (1 + \dfrac{l_1}{n})\ k$,

And $F_2 = -k_2 l_2 = -k(l_2 + l_2)\ \therefore\ k_2(n + 1)k$

4. $m = 100\ g$, $A(t) = \frac{A}{2}$, $t = 100 \times 2 = 200$ s, $A(t) = A^{-bt/2m}$

5. $v = \pm\omega\sqrt{4A^2 - 3y^2}$, $v_{new} = \pm\omega\sqrt{A_1^{\ 2} - y_1^{\ 2}}$ As $v_{new} = 2\ v$, $2\sqrt{A^2_{\ new} - y^2} = \sqrt{A^2_{\ new} - y^2}$, $4(A^2 - y^2) = A^2_{\ new} - y^2$ $\therefore$ $A^2_{\ new}\ A4^2 - 4y^2 + y,$

$A_{new} = \sqrt{4A^2 - 3y^2}$

6. $v = \omega\sqrt{A^2 - y^2}$, $a = -\omega^2 y$, $T = \frac{2\pi}{\omega}$, $a^2T^2 + 4\pi^2v^2 = 4\pi^2\omega^2A^2 = \text{Constant}.$

7. $T - mg\cos\theta = mv^2/L$ $\therefore$ $T = mg\cos\theta + mv^2/L$

$T = T_{max}$, when $\cos\theta = 1$ and v is maximum

$v^2_{\ max} = 2\ hg = 2g\ L\ \frac{\theta_0^{\ 2}}{2}$, $v^2_{\ max} = 2\ hg = 2g\ L\ (1 - \cos\theta_1),$

$= 2g\ L\ (\sin^2\frac{\theta_0}{2})\ (\because\ \sin^2\theta = \frac{1-\cos^2\theta}{2}) = 2g\ L\ \frac{\theta_0^{\ 2}}{2}$

$gL\left(\frac{A}{L}\right)^2$ $T_{max} = mg\left[1 + \left(\frac{A}{L}\right)^2\right]$

8. $y_1 = 10\sin(3\pi t + \frac{\pi}{4})$, $A_1 = 10$, $\omega_1 = 3\pi \Rightarrow T_1 = \frac{2}{3}$ s

$y_2 = 5\ (\sin 3\pi t + \sqrt{3}\cos 3\pi t) = A_2\cos\phi\ \sin 3\pi t + A_2\sin\phi\ \cos 3\pi t$

$y_2 = A_2\sin(3\pi t + \phi)$, $A_2 = \sqrt{(5)^2 + (5\sqrt{3})^2} = 10$

$\omega_2 = 3\pi$, $T_2 = \frac{2}{3}$ s, And $\frac{A_1}{A_2} = 1$

9. $PE = \frac{1}{2}ky^2$. Total mechanical energy $E = K + U$ $\therefore$ $K = E - U$

10. $v_1 = \omega\sqrt{A^2 - y_1^{\ 2}}$, $v_2 = \omega\sqrt{A^2 - y_2^{\ 2}}$, $v_1^{\ 2} - v_2^{\ 2} = \omega^2(y_2^{\ 2} - y_1^{\ 2})$

$T = \frac{2\pi}{\omega}$

CHAPTER 8

1. Differentiate wave equation $y = A \sin(\omega t - kx)$ w.r.t 't', the instantaneous velocity of a particle at time 't' will be, $v_p = \frac{dy}{dt} = A\omega \cos(\omega t - kx)$.

Now, wave speed $v = \omega/k$

Slope of wave at $x = \frac{dy}{dx} = -kA\cos(\omega t - kx)$

From above all three equations, $\frac{v_P}{v} = -\frac{dy}{dx}$

2. Speed of P wave $v_P = \frac{d}{t}$, Speed of S wave $v_S = \frac{d}{t+240}$

($\because$ 4 min = 60 × 4 = 240 s)

By solving these two equations $t = 240$ s

Now, substitute value of t and v_P in equation $v_P = \frac{d}{t}$ and find out d.

3. A = 10 m, $x_1 = 2$ m, $t_1 = 2$ s and $y_1 = 5$ m, $x_2 = 16$ m, $t_2 = 8$ s and $y_2 = 5\sqrt{3}$ m

Now, substitute these values in equation $y_1 = a \sin(\omega t_1 - kx_1)$

$\omega - k = \frac{\pi}{12}$ (1)

From equation $y_2 = A\sin(\omega t_2 - kx_2)$ you will get,

$\omega - 2k = \frac{\pi}{24}$ (2)

Subtracting equation (2) from equation (1)

$k = \frac{\pi}{24}$ rad/m, substitute value of k in equation (1), $\omega = \pi/8$ rad/s

4. $y = 3 \sin((3.14)x - (314)t)$ differentiate equation w.r.t. 't'

$v = \frac{dy}{dx} = (3)(314)\cos((3.14)x - (314)t)$

$\therefore$ Max speed of particle = (3) (314) = 9.4 m s^{-1}. Differentiate above equation w.r.t. 't'.

$$a = \frac{dv}{dt} = -(3)(314)(314)\ \sin\ (\ (3.14)x - (314)t)$$

Now put $x = 6$ cm and $t = 0.11$ s,

$a = -\ (3)\ (314)^2\ \sin\ (6\pi - 11\pi) = (-3)\ (314)^2\ \sin\ (-5\pi) = 0.$

5. $T_1 = 0. + 273 = 273$ K, $\lambda_1 = 1.32$ m, $T_2 = 27 + 273 = 300$ K, $\lambda_2 = ?$

Now, $\dfrac{v_1}{v_2} = \sqrt{\dfrac{T_1}{T_2}}$

$\therefore\ \dfrac{\lambda_1}{\lambda_2} = \sqrt{\dfrac{T_1}{T_2}}\ \ (\because\ v = f\lambda)$

Substitute the values in above equation, $\lambda_2 = 1.384$ m,

Increase in the wavelength $\Delta\lambda = \lambda_2 - \lambda_1 = 0.064$ m

6. $T_0 = 1200 + 273 = 1473$ K, $\rho_0 = 16\ \rho_H$, $T_H = ?$ Now, $v_0 = v_H$

$$\therefore\ \sqrt{\frac{\gamma R T_0}{\rho_0 V}} = \sqrt{\frac{\gamma R T_H}{\rho_H V}}\quad \therefore\ T_H = T_0 \times \frac{\rho_H}{\rho_0} = 1473 \times \frac{1}{16} = 92.06\ \text{K}$$

$\therefore\ T_H = 92.06 - 273 = -\ 180.94^{\circ}$C

7. Wave speed is same in all parts of the wire as the medium (wire) is same

$\therefore\ v = f_1\lambda_1 = f_2\lambda_2 = f_3\lambda_3$

Each section of wire is oscillating with fundamental frequency ($f = 2$L)

$\therefore\ f_1\ (2L_1) = f_2\ (2L_2) = f_3\ (2L_3)$, Now, put $f_1 : f_2 = 1 : 2$ and $f_1 : f_3 = 1 : 3$

in above equation and determine L_1, L_2 and L_3.

8. $\mu = 0.05$ g/cm, $f_n = 420$ Hz, $f_{n+1} = 490$ Hz, T = 490 N

Suppose the wire vibrates at 420 Hz in its nth harmonic and at 490 Hz in its (n + 1)th harmonic. According to $f = \dfrac{n}{2L}\sqrt{\dfrac{T}{\mu}}$

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \quad (1)\ \text{and} \qquad f_{n+1} = \frac{n+1}{2L}\sqrt{\frac{T}{\mu}} \quad (2)$$

Taking the ratio,

$$\frac{f_{n+1}}{f_n} = \frac{n+1}{n} \qquad \therefore\ n = 6 \qquad \text{(by putting value of } f_n \text{ and } f_{n+1})$$

$$420 = \frac{6}{2L}\sqrt{\frac{450}{5 \times 10^{-3}}} = \frac{900}{L}\quad \therefore\ L = \frac{900}{420} = 2.1\ \text{m}$$

9. L = 100 cm, f_n = 300 Hz, f_{n+1} = 400 Hz, 2A = 10 cm

Now, $f_{n+1} - f_n = (n+1)f_1 - nf_1$, $\therefore f_1 = 100$ Hz

$\pi = \frac{2L}{1} = 200$ cm, $\therefore k = \frac{2\pi}{\lambda} = \frac{\pi}{100}$ rad/cm

$\omega = 2\pi f_1 = 2\pi(100)$ rad/s

Equation of stationary wave, $y = -10 \sin(\frac{\pi}{100}x) \cos(200\pi)t$ cm

10. When the car is moving towards the listener, $f_{L_1} = \left(\frac{v+0}{v-v_S}\right) f_s$

When the car is moving away from the listener, $f_{L_2} = \left(\frac{v+0}{v+v_S}\right) f_S$

$$\therefore\ f_{L_1} - f_{L_2} = \left(\frac{v}{v-v_S} - \frac{v}{v+v_S}\right) f_S$$

Substitute, v = 340 m/s, v_S = 15 m/s and f_S = 500 Hz in above equation.

$f_{L_1} - f_{L_2}$ = 44.2Hz

11. f_S = 600 Hz, v = 340 m/s, v_L = 10 m s^{-1}

When the engine is moving with the speed 10 m s^{-1} towards the hill, we can consider its image moving in opposite direction. Listener is sitting in the engine and engine is moving towards the hill. Hence, direction of v_L is along L to S and direction of v_S is from S to L.

$$\therefore f_L = \frac{v+v_L}{v-v_S} \times f_S = \frac{340+10}{340-10} \times 600 = 700 \text{ Hz}$$

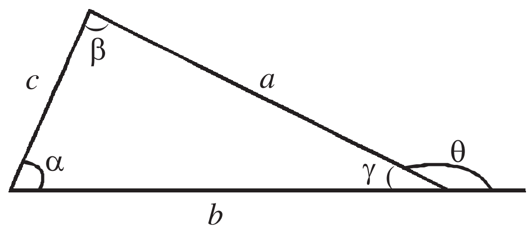
APPENDIX

SINE AND COSINE RULES

(i) $\frac{\sin\ \alpha}{a} = \frac{\sin\ \beta}{b} = \frac{\sin\ \gamma}{c}$

(ii) $c^2 = a^2 + b^2 - 2\ ab\cos\ \gamma$

(iii) Exterior angle, $\theta = \alpha + \beta$



TRIGONOMETRIC IDENTITIES

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(iii) $1 + \cot^2\theta = \mathrm{cosec}^2\theta$

(iv) $\sec^2\theta - \tan^2\theta = 1$

(v) $\mathrm{cosec}^2\theta - \cot^2\theta = 1$

(vi) $\sin 2\theta = 2\sin\theta\cos\theta$

(vii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

(viii) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

(ix) $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

(x) $\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right)$

(xi) $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$

(xii) $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$

Values of sine and cosine for special angles :

Function	0° 0 rad	30° $\frac{\pi}{6}$ rad	45° $\frac{\pi}{4}$ rad	60° $\frac{\pi}{3}$ rad	90° $\frac{\pi}{2}$ rad	180° π rad	270° $\frac{3\pi}{2}$ rad	360° 2π rad
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0

Quadratic Formula :

If $ax^2 + bx + c = 0$, then, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Formulae of Log :

1. If $log\ a = x$, then $a = 10^x$
2. $log(ab) = log(a) + log(b)$
3. $log\left(\dfrac{a}{b}\right) = log(a) - log(b)$
4. $log(a^n) = n\ log\ a$
5. $log_a a = 1$
6. $ln\ a = log_e{}^a = 2.303\ log_{10} a$

Important Expansions :

1. Binomial Expansion $(1 \pm x)^n = 1 \pm nx + \dfrac{n(n-1)x^2}{2!} + ...\ (x < 1)$

   $(1 \pm x)^{-n} = 1 \mp nx + \dfrac{n(n+1)x^2}{2!}\ ......\ (x < 1)$

2. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^2}{3!} + ...$ when $x << 1$, then $e^x = 1 + x$

3. $ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^2}{3} + ...(|\ x\ | < 1)$

   when $x << 1$, then $ln(1 \pm x) = \pm x$.

4. Trigonometric Expansion (θ in radian)

   (i) $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + ...$

   (ii) $\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + ...$

   (iii) $\tan\theta = \theta + \dfrac{\theta^3}{3} + \dfrac{\theta^5}{15} + ...$

   If θ is very small, then $sin\theta \approx \theta$; $cos\theta \approx 1$ and $tan\theta \approx \theta$ rad

y	$\dfrac{dy}{dx}$	y	$\dfrac{dy}{dx}$
x^n	nx^{n-1}	$sec\ x$	$\sec x \tan x$
$sin\ x$	$cos\ x$	$cosec\ x$	$-cosec\ x\ cot\ x$
$cos\ x$	$-sin^{-2}\ x$	lnx	$\dfrac{1}{x}$
$cot\ x$	$-cosec^2\ x$	$tan\ x$	$sec^2\ x$
$cos\ kx$	$-k\ sin\ x$	e^x	e^x
$sin\ kx$	$k\ cos\ x$	a^x	$a^x\ ln\ a$

Working rules of derivatives :

(1) $\dfrac{d}{dx}(k) = 0$ (where, k is a constant)

(2) $\dfrac{d}{dx}(x) = 1$

(3) $\dfrac{d}{dx}(ky) = k\dfrac{dy}{dx}$ (where k is a constant)

(4) $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

(5) If $y = u \pm v$, then $\dfrac{dy}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$

(6) If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} \pm v\dfrac{du}{dx}$

(7) If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Integrals of Some Standard Functions :

$f(x)$	$F(x) = \int f(x)dx$	$f(x)$	$F(x) = \int f(x)dx$
x^n $(n \neq + 1)$	$\frac{x^{n+1}}{n+1} + c$	$(ax + b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
$\frac{1}{x}$	$ln\ x + c$	$sin\ x$	$-cos\ x + c$
e^x	$e^x + c$	$cos\ x$	$sin\ x + c$
e^{kx}	$\frac{1}{k}e^{kx} + c$	$sin\ kx$	$-\frac{1}{k}cos\ x + c$
a^x	$\frac{a^x}{lna} + c$	$cos\ kx$	$\frac{1}{k}sin\ kx + c$

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REFERENCE BOOKS

1. PHYSICS, Part 1 and 2, Std. XI, GSBST
2. PHYSICS, Part 1 and 2, Std. XI, NCERT
3. Fundamentals of PHYSICS by Halliday, Resnick and Walker
4. University Physics by Young, Zemansky and Sears
5. CONCEPTS OF PHYSICS by H. C. Verma
6. Advanced PHYSICS by Tom Duncan
7. Advanced LEVEL PHYSICS by Nelkon and Parker
8. FUNDAMENTAL UNIVERSITY PHYSICS by Alonso and Finn
9. COLLEGE PHYSICS by Weber, Manning, White and Weygand
10. PHYSICS FOR SCIENTIST AND ENGINEERS by Fishbane, Gasiorowicz, Thornton
11. PHYSICS by Cutnell and Johnson
12. COLLEGE PHYSICS by Serway and Faughn
13. UNIVERSITY PHYSICS by Ronald Reese
14. CONCEPTUAL PHYSICS by Hewitt
15. PHYSICS FOR SCIENTIST AND ENGINEERS by Giancoli
16. Heat Transfer by Holman

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LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5566	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6345	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7769	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9843	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Antilogarithms

	0	1	2	3	4	5	6	7	8	9	Mean Difference 1	2	3	4	5	6	7	8	9
00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1289	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Antilogarithms

	0	1	2	3	4	5	6	7	8	9	Mean Difference 1	2	3	4	5	6	7	8	9
50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
82	6607	6622	6637	6653	6668	6683	6699	6715	6730	6745	2	3	5	6	8	9	11	12	14
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	12	14	16
90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

NATURAL SINES

Degree	0' 0.0°	6' 0.1°	12' 0.2°	18' 0.3°	24' 0.4°	30' 0.5°	36' 0.6°	42' 0.7°	48' 0.8°	54' 0.9°	Mean Deifferences 1'	2'	3'	4'	5'
0	.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	.0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	.0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	.0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	.1045	1063	1080	1197	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	.1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	.1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	.1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	.1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	.2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	.2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	.2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	.2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	.2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	.3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	.3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	.3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	.3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	.3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	.4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	.4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	.4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	.4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	.4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	.5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	.5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	.5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	.5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	.5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	.6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	.6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	.6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	.6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	.6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	.6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	.6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES

Degree	0' 0.0°	6' 0.1°	12' 0.2°	18' 0.3°	24' 0.4°	30' 0.5°	36' 0.6°	42' 0.7°	48' 0.8°	54' 0.9°	Mean Deifferences 1'	2'	3'	4'	5'
45	.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	.7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	.7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	.7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	.7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	.7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	.7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	.7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	.8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	.8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	.8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	.8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	.8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	.8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	.8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	.8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	.8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	.8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	.9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	.9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	.9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	.9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	.9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	.9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	.9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	.9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	.9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	.9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	.9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	.9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	.9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	.9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	.9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	.9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	.9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	.9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	.9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	.9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	.9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	.9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	.9998	9999	9999	9999	9999	1.000	1.000	1.000	1.000	1.000	0	0	0	0	0

NATURAL TANGENTS

Degree	0	6	12	18	24	30	36	42	48	54	Main Deifferences				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	.0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	.0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	.0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	.0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	.1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	.1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	.1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	.1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	.1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	.2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	.2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	.2493	2512	2530	2549	2568	2686	2605	2623	2642	2661	3	6	9	12	16
15	.2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	.2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	.3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	.3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	.3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	.3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	.4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	.4245	4265	8286	4307	4327	4748	4369	4390	4411	4431	3	7	10	14	17
24	.4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	.4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	.4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	.5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	.5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	.5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	.6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	.6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	.6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	.6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	.7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	.7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	.7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	.7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	.8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	.8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	.9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	.9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

NATURAL TANGENTS

Degree	0	6	12	18	24	30	36	42	48	54	Main Deifferences				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1.0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1.0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1.1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1.1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1.2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1.2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1.3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1.3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1.4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1.5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1.6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1.7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1.8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1.8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1.9626	9711	9797	9883	9970	2.0057	2.0145	2.0233	2.0323	2.413	15	29	44	58	73
64	2.0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2.1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2.2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2.3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2.4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2.6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2.7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2.9042	9208	9375	9544	9714	9887	3.0061	3.0237	3.0415	3.0595	29	58	87	116	145
72	3.0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3.2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3.4824	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3.7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4.0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	213	267
77	4.3315	3662	4015	4374	4737	5107	5483	5864	6252	6646	Mean Differences no longer sufficiently accurate				
78	4.7046	7453	7867	8288	8716	9152	9594	5.0045	5.0504	5.0970					
79	5.1446	1929	2422	2924	3465	3955	4486	5026	5578	6140					
80	5.6713	7297	7894	8502	9124	9758	6.0405	6.1066	6.1742	6.2432					
81	6.3138	3859	4596	5350	6122	6912	7720	8548	9395	7.0264					
82	7.1154	2066	3002	3962	4997	5958	6996	8062	9158	8.0285					
83	8.1443	2636	3863	5126	6427	7769	9152	9.0579	9.2052	9.3572					
84	9.514	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20					
85	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95					
86	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46					
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27					
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08					
89	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					